부정적분 200題

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(1) 각변환

$\sin(-\theta) = -\sin\theta$　　$\cos(-\theta) = \cos\theta$　　$\tan(-\theta) = -\tan\theta$

$\sin(\pi+\theta) = -\sin\theta$　　$\cos(\pi+\theta) = -\cos\theta$　　$\tan(\pi+\theta) = \tan\theta$

$\sin(\pi-\theta) = \sin\theta$　　$\cos(\pi-\theta) = -\cos\theta$　　$\tan(\pi-\theta) = -\tan\theta$

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$　　$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$　　$\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$　　$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$　　$\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

$\sin\left(\frac{3}{2}\pi+\theta\right) = -\cos\theta$　　$\cos\left(\frac{3}{2}\pi+\theta\right) = \sin\theta$　　$\tan\left(\frac{3}{2}\pi+\theta\right) = -\cot\theta$

$\sin\left(\frac{3}{2}\pi-\theta\right) = -\cos\theta$　　$\cos\left(\frac{3}{2}\pi-\theta\right) = -\sin\theta$　　$\tan\left(\frac{3}{2}\pi-\theta\right) = \cot\theta$

(2) 덧셈정리

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha\pm\beta) = \dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta} \quad (1\mp\tan\alpha\tan\beta \neq 0)$

(3) 배각공식

$\sin2\alpha = 2\sin\alpha\cos\alpha$

$\cos2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\tan2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} \quad (1-\tan^2\alpha \neq 0)$

$\sin3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\cos3\alpha = 4\cos^3\alpha - 3\cos\alpha$

$\tan3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} \quad (1-3\tan^2\alpha \neq 0)$

(4) 반각공식

$\sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$

$\cos^2\dfrac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$

$\tan^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{1+\cos\alpha}$

(5) 곱을 합 또는 차로 고치는 공식

$\sin\alpha\cos\beta = \frac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$

$\cos\alpha\sin\beta = \frac{1}{2}\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\}$

$\cos\alpha\cos\beta = \frac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$

$\sin\alpha\sin\beta = -\frac{1}{2}\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}$

(6) 합 또는 차를 곱으로 고치는 공식

$\sin A+\sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

$\sin A-\sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A+\cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A-\cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

(7) 삼각함수의 합성

$a\sin\theta+b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\alpha) = \sqrt{a^2+b^2}\cos(\theta-\beta)$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right),\ \beta = \tan^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{2}-\alpha$

(8) 삼각함수 항등식

$\sin^2\theta+\cos^2\theta = 1$

$\tan^2\theta+1 = \sec^2\theta$

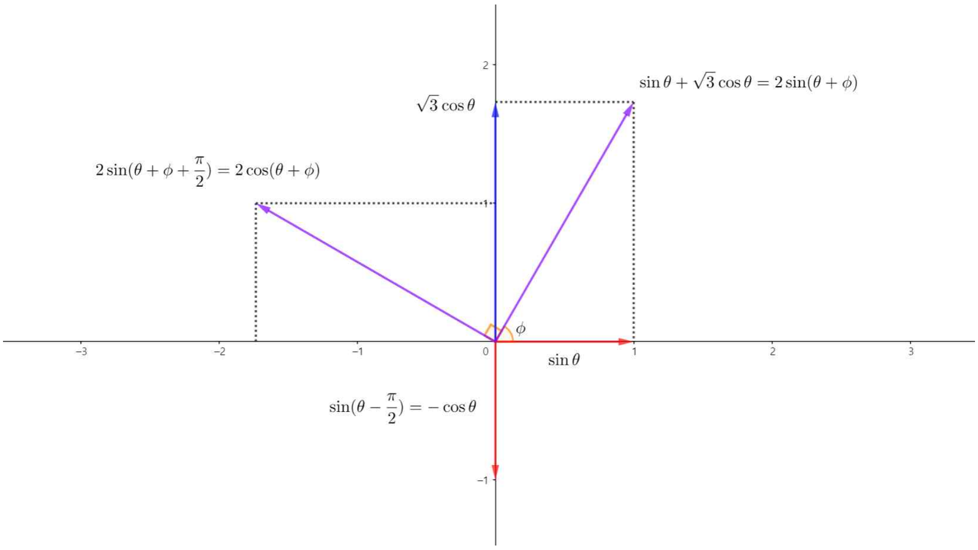
$\cot^2\theta+1 = \csc^2\theta$

(9) 부정적분과 미분

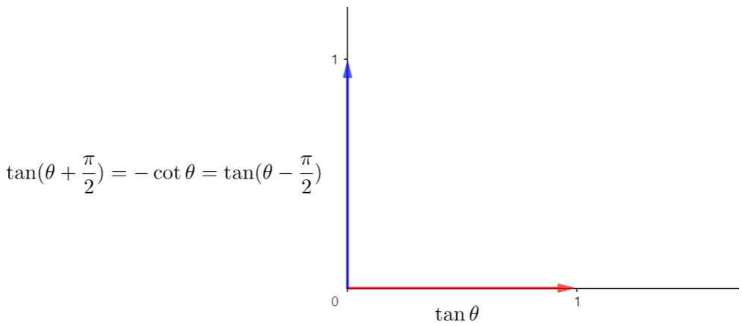
① $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

② $\int\left(\frac{d}{dx}f(x)\right)dx = f(x)+C$

(10) 삼각함수 위상자(Phasor)



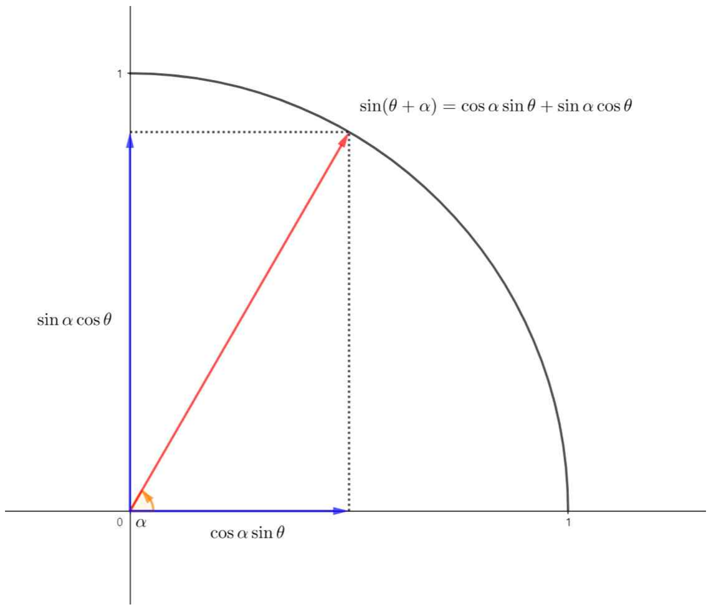
- x축의 양의 방향을 sin축, y축의 양의 방향을 cos축으로 설정하여 삼각함수의 위상을 벡터로 표현한다.
- 각이 더해질 경우 위상자는 길이는 유지된 채 반시계방향으로 회전한다.
- 위상자가 표현하는 삼각함수의 계수는 그 위상자의 길이로 표현된다.
- 서로 다른 두 위상자를 벡터합하면 이는 각 위상자가 표현하는 삼각함수의 합성과 같다. 가령, 위 사진에서 $\sin\theta + \sqrt{3}\cos\theta = 2\sin(\theta+\phi)$이고 $\phi = \frac{\pi}{3}$이다.
- $\sin\theta$ 위상자를 시계방향으로 $\frac{\pi}{2}$ 만큼 회전시키면 $\sin\left(\theta - \frac{\pi}{2}\right)$이며, 이는 $-\cos$축이므로 $\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$이다.
- sin을 csc로, cos을 sec로 바꾸면 csc와 sec에 대한 각변환이 가능하나 덧셈정리와 합성은 성립하지 않는다. 즉, $\csc\theta + \sqrt{3}\sec\theta = 2\csc(\theta+\phi)$는 성립하지 않는다.



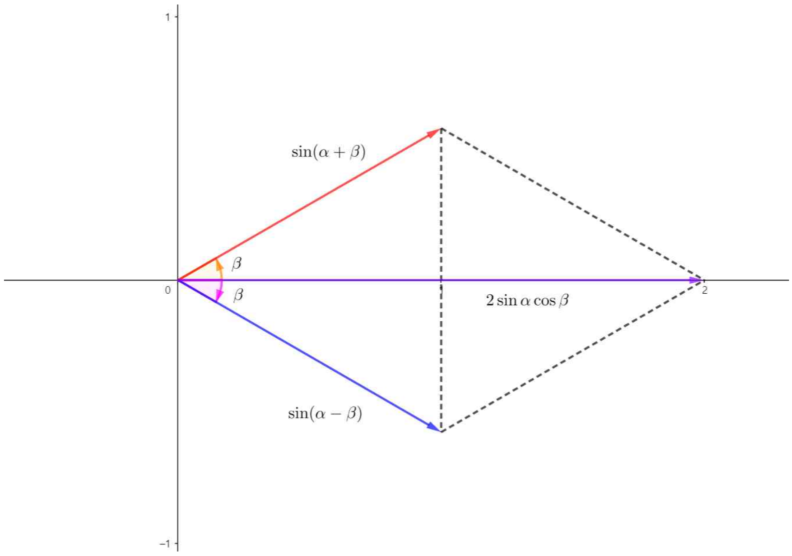
- tan와 $-\cot$의 경우 위와 같이 xy평면의 제 1사분면만을 이용하여 도시할 수 있다. 이 경우 주기가 π이므로 시계방향으로 회전하는 경우 돌아간 각도를 $\frac{3}{2}\pi$가 아닌 $\frac{\pi}{2}$로 본다.

(11) 삼각함수 위상자의 활용

- 삼각함수 위상자를 사용하면 (1), (2), (5), (6), (7)의 공식들은 모두 증명 가능하다. 이에 대한 예시로 (2)와 (5)의 첫 번째 공식의 증명을 제시해 놓는다.



그림과 같은 단위원에서 $\sin(\theta+\alpha)$는 $\sin\theta$축에서 길이 1인 위상자가 각도 α만큼 회전한 것이다. 따라서 위상자의 종점에서 sin축, cos축에 각각 수선의 발을 내리면 원점을 시점으로 하고 두 수선의 발을 종점으로 하는 두 위상자의 길이는 각각 $\cos\alpha$, $\sin\alpha$이다. 즉 처음의 $\sin(\theta+\alpha)$ 위상자(빨간색)를 sin축 성분과 cos축 성분의 두 위상자(파란색)로 분해할 수 있고 이들의 길이는 각각 $\cos\alpha$, $\sin\alpha$이므로, $\sin(\theta+\alpha) = \cos\alpha\sin\theta + \sin\alpha\cos\theta$가 성립한다.



그림과 같이 기준각이 α인 위상 평면에서 $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$는 길이가 1인 $\sin\alpha$축 위상자가 각각 β, $-\beta$만큼 회전한 것이다. 따라서 이들을 합성한 보라색 위상자는 $\sin\alpha$축으로 길이가 $2\cos\beta$인 위상자이므로 $\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin\alpha\cos\beta$이고 증명이 완료되었다.

이와 비슷한 방법으로 (1), (2), (5), (6), (7)의 공식들을 모두 증명할 수 있다.

(12) 부정적분의 기본 성질

① $\int kf(x)dx = k\int f(x)dx \quad (k \in \mathbb{R})$

② $\int \{f(x) \pm g(x)\}dx = \int f(x)dx \pm \int g(x)dx$ (복부호동순)

(13) 치환적분법

① $g(x) = t$일 때, $\int f(g(x))g'(x)dx = \int f(t)dt$

② $\int f(x)dx = F(x) + C$일 때, $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

③ $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + C$

④ $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$

(14) 바이어슈트라스 치환 (Weierstrass Substitution)

- 바이어슈트라스 치환은 삼각함수의 유리 적분을 유리식의 적분으로 바꿔주는 치환법이다.

$$\tan\frac{x}{2} = t$$

$$\sin x = \frac{2t}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \ \tan x = \frac{2t}{1-t^2}$$

(15) 오일러 치환 (Euler's Substitution)

- 오일러 치환은 유리 이변수 함수 R에 대하여 다음과 같은 부정적분을 계산하기 위한 치환법이다.

$$\int R(x, \ \sqrt{ax^2+bx+c})dx$$

[1] 제 1종 오일러 치환

$a > 0$일 때,

$$\sqrt{ax^2+bx+c} = \pm x\sqrt{a} + t,$$

$$x = \frac{c-t^2}{\pm 2t\sqrt{a} - b}$$

와 같이 치환한다.

[2] 제 2종 오일러 치환
$c>0$일 때,

$$\sqrt{ax^2+bx+c}=xt\pm\sqrt{c},$$

$$x=\frac{\pm 2t\sqrt{c}-b}{a-t^2}$$

와 같이 치환한다.

[3] 제 3종 오일러 치환
방정식 $ax^2+bx+c=0$이 두 실근 α, β를 가질 때,

$$\sqrt{ax^2+bx+c}=\sqrt{a(x-\alpha)(x-\beta)}=(x-\alpha)t,$$

$$x=\frac{\alpha\beta-\alpha t^2}{a-t^2}$$

와 같이 치환한다.

(16) 부정적분 공식
1. $\int dx=x+C$
2. $\int a dx=ax+C$ $(a\in\mathbb{R})$
3. $\int x^n dx=\frac{1}{n+1}x^{n+1}+C$ $(-1\neq n\in\mathbb{R})$
4. $\int \frac{1}{x}dx=\ln|x|+C$
5. $\int e^x dx=e^x+C$
6. $\int a^x dx=\frac{a^x}{\ln a}+C$ $(0<a\neq 1)$
7. $\int \ln x dx=x\ln x-x+C$
8. $\int \sin x dx=-\cos x+C$
9. $\int \cos x dx=\sin x+C$
10. $\int \tan x dx=\ln|\sec x|+C$
11. $\int \csc x dx=\ln|\csc x-\cot x|+C$
12. $\int \sec x dx=\ln|\sec x+\tan x|+C$

13. $\int \cot x dx = \ln|\sin x| + C$

14. $\int \sec^2 x dx = \tan x + C$

15. $\int \csc^2 x dx = -\cot x + C$

16. $\int \sec x \tan x dx = \sec x + C$

17. $\int \csc x \cot x dx = -\csc x + C$

18. $\int \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln\left|\frac{x+a}{x+b}\right| + C$

19. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$

21. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

22. $\int \sin^{-1} dx = x \sin^{-1} x + \sqrt{1-x^2} + C$

23. $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1-x^2} + C$

24. $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

(17) 헤비사이드 법 (부분분수 분해)

$\frac{g(x)}{f(x)} = \frac{g(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \sum_{i=1}^{n} \frac{b_i}{x-a_i} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \cdots + \frac{b_n}{x-a_n}$ 일 때

$\frac{f(x)}{(x-a_i)} = h_i(x)$라 하면 $b_i = \frac{g(a_i)}{h_i(a_i)}$가 성립한다. $(1 \le i \le n)$

(18) 부분적분법

$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

(19) 삼각함수의 거듭제곱의 부정적분 공식 (Reduction Formula)

$n \in \mathbb{N} - \{1\}$,

$\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$

$\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx$

$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$

※ 부정적분은 원시함수들의 집합이므로 예시 답과 다르다고 틀린 것이 아닙니다. Wolfram Alpha 등을 이용하여 미분했을 때 맞게 나온다면 정답입니다. 다만 모범 답안에는 정석적인 방법론으로 최대한 간단하게 풀이하였으니 모범 답안과 다르게 나온 경우에는 답안의 풀이도 한 번 읽어보시길 바랍니다.

※※ 쌍곡선함수($\sinh x$, $\cosh x$, $\tanh x$)와 그 역함수는 등장하지 않으며, 역쌍곡선함수의 경우 $\tanh^{-1}x$라는 표기 대신 $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ 등으로 나타내었습니다. 적분 결과 또는 피적분함수에 역삼각함수가 등장하는 경우 고등학교 교육과정을 벗어나는 것이기는 하나, (16)의 19, 21번 공식을 사용하는 수준에서 모두 해결 가능합니다.

역삼각함수가 등장하는 문항은 다음과 같습니다. (문제 번호를 빨간색으로 표시했습니다.)

25, 29, 43, 65, 68, 69, 79, 83, 85, 86, 90, 103, 104, 105, 106, 108, 109, 111, 143, 144, 146, 151, 153, 159
161, 164, 165, 166, 168, 169, 173, 174, 175, 176, 178, 179, 180, 182, 183, 184, 185, 191, 193, 194, 196, 197, 199

[1~200] 다음 부정적분을 구하시오.

1. $\int \frac{x^3-2x+1}{\sqrt[3]{x}}dx$

2. $\int \cos^2\frac{x}{2}dx$

3. $\int 2^{x-1}dx$

4. $\int \tan x dx$

5. $\int \sin x\cos^2 x dx$

6. $\int \sec^2 x dx$

7. $\int \frac{3}{3x+2}dx$

8. $\int x\sqrt{x^2+1}\,dx$

9. $\int \frac{x^2}{x-2}dx+\int \frac{3x-10}{x-2}dx$

10. $\int \csc x(\sin x+\cot x)dx$

11. $\int 2^{2x}dx$

12. $\int \frac{e^{2x}-4}{e^x-2}dx$

13. $\int \sin 3x\cos 4x dx$

14. $\int \sin x\cos 3x dx$

15. $\int \cos 2x \cos 6x dx$

16. $\int \sin^4 x dx$

17. $\int \cos^4 x dx$

18. $\int \sin^6 x \cos^3 x dx$

19. $\int \sin^3 x \sec^2 x dx$

20. $\int \sin^4 x \cos^4 x dx$

21. $\int \sin^2 x \cos^5 x dx$

22. $\int \sin^3 x \cos^5 x dx$

23. $\int \sin^2 x \cos^4 x dx$

24. $\int \sin^4 x \cos^2 x dx$

25. $\int \sqrt{1-4x^2}\, dx$

26. $\int \frac{1}{x^2\sqrt{x^2+4}} dx$

27. $\int \frac{1}{(x+1)\sqrt{x^2+2x+2}} dx$

28. $\int \frac{x^2+2x-2}{x^3-4x} dx$

29. $\int \frac{1}{x^4-1} dx$

30. $\int (x^2+2x+2)^5 (x+1) dx$

31. $\int (x^2-1)\sqrt{x^3-3x}\, dx$

32. $\int (1+\sin x)^2 \cos x dx$

33. $\int \sin x \cos 2x dx$

34. $\int \frac{e^x}{\sqrt{e^x+1}} dx$

35. $\int \frac{2x+1}{x^2+x+1} dx$

36. $\int \frac{x+1}{x^3-1} dx$

37. $\int \tan 2x dx$

38. $\int \frac{1}{x \ln x} dx$

39. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

40. $\int (2+3x)\sqrt{1+2x}\, dx$

41. $\int \frac{x-1}{\sqrt{x+1}} dx$

42. $\int \frac{x^3}{\sqrt{1-x^2}} dx$

43. $\int \frac{1}{1+\cos^2 x} dx$

44. $\int \frac{x}{(2x+1)^3} dx$

45. $\int \frac{1}{x^3(x+1)} dx$

46. $\int \frac{1}{1+\cos x} dx$

47. $\int \frac{x}{\sqrt{1-x^2}} dx$

48. $\int \frac{e^x - 1}{e^x + 1} dx$

49. $\int \frac{1}{e^{2x}+1} dx$

50. $\int \frac{1}{e^{2x}+e^x} dx$

51. $\int \frac{e^x \ln(e^x+1)}{e^x+1} dx$

52. $\int \frac{e^x(e^x+1)}{(e^x+3)^2} dx$

53. $\int \frac{\ln x}{x(\ln x+1)^2} dx$

54. $\int \frac{x^4}{(x-1)^3} dx$

55. $\int \frac{x^2}{(x^2-a^2)(x^2-b^2)} dx \quad (a \neq b)$

56. $\int x^3 \sqrt{x^2+a}\, dx$

57. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1} dx$

58. $\int \frac{1}{1+\tan x} dx$

59. $\int \frac{1}{\sin 2x - \sin x} dx$

60. $\int \frac{1}{\sin x} dx$

61. $\int \frac{1}{1-3\cos^2 x} dx$

62. $\int \frac{\sin x}{1+\sin x} dx$

63. $\int \sqrt{1+\sqrt{x}}\, dx$

64. $\int \sqrt{x+\sqrt{x^2+3}}\, dx$

65. $\int \frac{x}{\sqrt{5-4x-x^2}} dx$

66. $\int \frac{x^3-x-2}{x^3-x^2+x-1} dx$

67. $\int \frac{1}{x^2+1} dx$

68. $\int \frac{1}{ax^2+bx+c} dx \quad (b^2-4ac<0,\ a>0)$

69. $\int \frac{1}{x^2+x+1} dx$

70. $\int \frac{1}{3+5\cos x} dx$

71. $\int \frac{1}{4\sin x+3\cos x} dx$

72. $\int \frac{1}{\sqrt{(a^2-x^2)^3}} dx \quad (a>0)$

73. $\int \frac{1}{\sqrt{(a^2+x^2)^3}} dx \quad (a>0)$

74. $\int \sin^2 x dx$

75. $\int \sin^3 x dx$

76. $\int \tan^2 x dx$

77. $\int \tan^3 x dx$

78. $\int \sec^3 x dx$

79. $\int \frac{1}{4x^2+4x+2} dx$

80. $\int \frac{1}{\sqrt{x^2+1}} dx$

81. $\int \frac{1}{\sqrt{x^2-4}} dx$

82. $\int \frac{1}{\sqrt{x^2-2x+2}}\,dx$

83. $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

84. $\int \frac{4x}{(x-1)^2(x+1)}\,dx$

85. $\int \frac{2x^2-x+4}{x(x^2+4)}\,dx$

86. $\int \frac{1}{16x^4-1}\,dx$

87. $\int \frac{x^2+1}{(x-1)(x-2)(x-3)}\,dx$

88. $\int \frac{x+4}{x^3+3x^2-10x}\,dx$

89. $\int \sqrt{x^2+1}\,dx$

90. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

91. $\int \frac{1}{x(1+x^2)}\,dx$

92. $\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$

93. $\int \frac{1}{x\sqrt{1-x^2}}\,dx$

94. $\int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}}\,dx \quad (\alpha,\ \beta \in \mathbb{R})$

95. $\int \frac{1}{3\cos^2 x-\sin^2 x}\,dx$

96. $\int \frac{7x^3-13x^2-24x+24}{x^4-3x^3-10x^2+24x}\,dx$

97. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x^3}}\,dx$

98. $\int \frac{1}{x^2\sqrt{2x-x^2}}\,dx$

99. $\int \frac{1}{x+\sqrt{x^2-1}}\,dx$

100. $\int \frac{1}{(x-1)^2\sqrt{3+2x-x^2}}\,dx$

101. $\int \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}\,dx$

102. $\int \frac{x^4+81}{x(x^2+9)^2}\,dx$

103. $\int \frac{x^2 - x + 2}{x^3 - 1} dx$

104. $\int \sqrt{\frac{4 - x}{x}} dx$

105. $\int \sqrt{\frac{x}{1 - x^3}} dx$

106. $\int \sqrt{x} \sqrt{1 - x} dx$

107. $\int \sqrt{\frac{x - 2}{x - 1}} dx$

108. $\int \frac{\sqrt{x^2 - 25}}{x^3} dx$

109. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$

110. $\int \frac{1}{\sin^2 x \cos^2 x} dx$

111. $\int \frac{1}{(1 + x^2)^2} dx$

112. $\int \tan^6 x \sec^4 x dx$

113. $\int \tan^4 x \sec^4 x dx$

114. $\int \tan x \sec^5 x dx$

115. $\int \tan^5 x \sec^7 x dx$

116. $\int x \sec^2 x dx$

117. $\int x \sin(x + 2) dx$

118. $\int \ln(x^2 - x) dx$

119. $\int e^x \sin^2 \frac{x}{2} dx$

120. $\int \frac{2x + 7}{\sqrt{x^2 + 6x - 7}} dx$

121. $\int \frac{2x - 1}{x^2 - 2x - 3} dx$

122. $\int 4 \sin 2x \cos 2x \cos 4x dx$

123. $\int (2x + 3) e^x dx$

124. $\int x \ln x dx$

125. $\int x^2 e^x dx$

126. $\int x^2 \ln x dx$

127. $\int x \cos x dx$

128. $\int (\ln x)^2 dx$

129. $\int e^x \sin x dx$

130. $\int e^x \cos x dx$

131. $\int e^x \sin^2 x dx$

132. $\int x^n \ln x dx \quad (n \in \mathbb{Z})$

133. $\int \ln x dx$

134. $\int \ln(x+1) dx$

135. $\int \frac{1}{1+e^x} dx$

136. $\int \frac{x^3+x^2+1}{x^2(x^2+1)} dx$

137. $\int (x^2+1) e^x dx$

138. $\int \frac{1}{x\sqrt{x+1}} dx$

139. $\int \frac{x\sqrt{x}-1}{x-\sqrt{x}} dx$

140. $\int \ln(x+\sqrt{x^2+1}) dx$

141. $\int \left(x^2 - \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right) dx$

142. $\int \sin\sqrt{x}\, dx$

143. $\int \ln(x^2+1) dx$

144. $\int \frac{-2x}{\sqrt{1-x^4}} dx$

145. $\int e^{\sqrt{x}} dx$

146. $\int \frac{x^2}{\sqrt{a^2-x^2}} dx$

147. $\int \frac{1}{x\sqrt{a^2-x^2}} dx$

148. $\int \frac{1}{\sqrt{1+e^x}} dx$

149. $\int \frac{x}{\sqrt{1+x^4}}dx$

150. $\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx$

151. $\int \frac{x+2}{(x^2+1)(x-1)^3}dx$

152. $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx$

153. $\int \frac{\tan^{-1}x}{1+x^2}dx$

154. $\int \frac{4}{x^2\sqrt{4-x^2}}dx$

155. $\int 3\sec^4 3x dx$

156. $\int \cot^3 x dx$

157. $\int \csc^4 x dx$

158. $\int \tan^2 3x dx$

159. $\int \frac{3x^4-3x^3-x^2-17x-2}{(x-3)(x^2+1)^2}dx$

160. $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

161. $\int \frac{1}{x-\sqrt{1-x^2}}dx$

162. $\int \frac{x^2+3}{x^2-1}dx$

163. $\int 13e^{2x}\cos 3x dx$

164. $\int \frac{x^4-2x^2+4x+1}{x^3+4x}dx$

165. $\int \frac{2x^2-x+4}{x^3+4x}dx$

166. $\int \frac{1}{(x+1)(x^2+1)}dx$

167. $\int \frac{x^2-2x-2}{x^3-1}dx$

168. $\int \sqrt{a^2-x^2}dx \quad (a>0)$

169. $\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}dx$

170. $\int \frac{1}{\sin(x-a)\sin(x-b)}dx$

171. $\int x\sec^2 x\tan x\,dx$

172. $\int e^{-x}\sin^2 2x\,dx$

173. $\int \sqrt{\tan x}\,dx$

174. $\int \sqrt[3]{\tan x}\,dx$

175. $\int \left(\frac{x+3}{\sqrt{4-x^2}}+\cot x[\ln(\sin x)]\right)dx$

176. $\int \left(\frac{4x^2}{x^2+9}+\tan^2 2x\right)dx$

177. $\int \frac{1}{(1+x)(2+x)^2(3+x)^3}dx$

178. $\int \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

179. $\int \frac{\tan x+\tan^3 x}{1+\tan^3 x}dx$

180. $\int \frac{1}{\cos 2x+3}dx$

181. $\int e^x\tan x(1-2\sec^2 x)dx$

182. $\int \sqrt{\tan x+1}\,dx$

183. $\int \sqrt{\tan x+2}\,dx$

184. $\int \frac{\tan^{-1}x}{x\sqrt{x^2+1}}\cdot\exp\left(-\frac{\tan^{-1}x}{x}\right)dx$ (단, $\exp x = e^x$)

185. $\int \cos^2(\tan^{-1}(\sin(\cot^{-1}x)))dx$

186. $\int \frac{\sec x-\tan x}{\sqrt{\sin^2 x-\sin x}}dx$

187. $\int \frac{\sec^2 x}{(\sec x+\tan x)^{5/2}}dx$

188. $\int \frac{\sec^2 x}{(\sec x+\tan x)^{9/2}}dx$

189. $\int \sin 2022x\cdot\sin^{2020}x\,dx$

190. $\int \frac{\sqrt{\cos 2x}}{\sin x}dx$

191. $\int \frac{1}{x\sqrt{x^2+4x-4}}dx$

192. $\int \frac{1}{x\sqrt{-x^2+x+2}}dx$

193. $\displaystyle\int \frac{x^2}{\sqrt{-x^2+3x-2}}dx$

194. $\displaystyle\int \frac{\sin^3(\theta/2)}{\cos(\theta/2)\cdot\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$

195. $\displaystyle\int \frac{\tan^4\theta}{1-\tan^2\theta}d\theta$

196. $\displaystyle\int \frac{x^2+1}{x^4+3x^3+3x^2-3x+1}dx$

197. $\displaystyle\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx$

198. $\displaystyle\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx$

199. $\displaystyle\int \left(\frac{\tan^{-1}x}{1+(x+1/x)\tan^{-1}x}\right)^2 dx$

200. $\displaystyle\int \frac{1}{\prod_{i=0}^{m}(x+i)}dx=\int \frac{1}{x(x+1)(x+2)\cdots(x+m)}dx \ \ (m\in\mathbb{N}\cup\{0\})$

[해설]

1. $t=\sqrt[3]{x}$, $dx=3t^2dt$

$$\int \frac{x^3-2x+1}{\sqrt[3]{x}}dx=\int 3t(t^9-2t^3+1)dt=\frac{3}{11}t^{11}-\frac{6}{5}x^5+\frac{3}{2}t^2+C=\frac{3}{11}x^{\frac{11}{3}}-\frac{6}{5}x^{\frac{5}{3}}+\frac{3}{2}x^{\frac{2}{3}}+C$$

2. $\displaystyle\int \cos^2\frac{x}{2}dx=\int \frac{1+\cos x}{2}=\frac{1}{2}x+\frac{1}{2}\sin x+C$

3. $\displaystyle\int 2^{x-1}dx=\frac{1}{\ln 2}2^{x-1}+C$

4. $\displaystyle\int \tan x\,dx=\int \frac{\sin x}{\cos x}dx=\ln|\sec x|+C$

5. $\displaystyle\int \sin x\cos^2 x\,dx=-\frac{1}{3}\cos^3 x+C$

6. $\displaystyle\int \sec^2 x\,dx=\tan x+C$

7. $\displaystyle\int \frac{3}{3x+2}dx=\ln|3x+2|+C$

8. $t=\sqrt{x^2+1}$, $tdt=xdx$

$$\int x\sqrt{x^2+1}\,dx=\int t^2dx=\frac{1}{3}t^3+C=\frac{1}{3}(x^2+1)^{\frac{3}{2}}+C$$

9. $\int \frac{x^2}{x-2}dx+\int \frac{3x-10}{x-2}dx=\int (x+5)dx=\frac{1}{2}x^2+5x+C$

10. $\int \csc x(\sin x+\cot x)dx=\int (1+\csc x\cot x)dx=x-\csc x+C$

11. $\int 2^{2x}dx=\frac{1}{2\ln 2}2^{2x}+C$

12. $\int \frac{e^{2x}-4}{e^x-2}dx=\int (e^x+2)dx=e^x+2x+C$

13. $\int \sin 3x\cos 4x\,dx=\frac{1}{2}\int (\sin 7x-\sin x)dx=\frac{1}{2}\cos x-\frac{1}{14}\cos 7x+C$

14. $\int \sin x\cos 3x\,dx=\frac{1}{2}\int (\sin 4x-\sin 2x)dx=\frac{1}{4}\cos 2x-\frac{1}{8}\cos 4x+C$

15. $\int \cos 2x\cos 6x\,dx=\frac{1}{2}\int (\cos 8x+\cos 4x)dx=\frac{1}{8}\sin 4x+\frac{1}{16}\sin 8x+C$

16. $\int \sin^4 x\,dx=\frac{1}{4}\int (1-\cos 2x)^2dx=\frac{1}{4}\int \left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$

$$=\int \left(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

17. $\int \cos^4 x\,dx=\frac{1}{4}\int (1+\cos 2x)^2dx=\frac{1}{4}\int \left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx$

$$=\int \left(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$$

18. $\int \sin^6 x\cos^3 x\,dx=\int \sin^6 x(1-\sin^2 x)\cos x\,dx=\int \sin^6 x\cos x\,dx-\int \sin^8 x\cos x\,dx$

$$=\frac{1}{7}\sin^7 x-\frac{1}{9}\sin^9 x+C$$

19. $\int \sin^3 x \sec^2 x dx = \int \frac{\sin x(1-\cos^2 x)}{\cos^2 x} dx = \int (\tan x \sec x - \sin x) dx = \sec x + \cos x + C$

20. $\int \sin^4 x \cos^4 x dx = \frac{1}{16}\int \sin^4 2x dx = \frac{1}{64}\int (1-\cos 4x)^2 dx$

$= \frac{1}{64}\int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right) dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$

21. $\int \sin^2 x \cos^5 x dx = \int \sin^2 x(1-\sin^2 x)^2 \cos x dx$

$= \int \sin^2 x \cos x dx - 2\int \sin^4 x \cos x dx + \int \sin^6 x \cos x dx = \frac{1}{7}\sin^7 x - \frac{2}{5}\sin^5 x + \frac{1}{3}\sin^3 x + C$

22. $\int \sin^3 x \cos^5 x dx = \int \sin^3 x(1-\sin^2 x)^2 \cos x dx$

$= \int \sin^3 x \cos x dx - 2\int \sin^5 x \cos x dx + \int \sin^7 x \cos x dx = \frac{1}{8}\sin^8 x - \frac{1}{3}\sin^6 x + \frac{1}{4}\sin^4 x + C$

23. $\int \sin^2 x \cos^4 x dx = \frac{1}{4}\int \sin^2 2x \cos^2 x dx = \frac{1}{16}\int (\sin 3x + \sin x)^2 dx$

$= \frac{1}{16}\int \left(\frac{1-\cos 6x}{2} + \frac{1-\cos 2x}{2} - \cos 4x + \cos 2x\right) dx$

$= \frac{1}{16}\int \left(1 + \frac{1}{2}\cos 2x - \cos 4x - \frac{1}{2}\cos 6x\right) dx = \frac{1}{16}x + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x + C$

24. $\int \sin^4 x \cos^2 x dx = \frac{1}{4}\int \sin^2 2x \sin^2 x dx = \frac{1}{16}\int (\cos 3x - \cos x)^2 dx$

$= \frac{1}{16}\int \left(\frac{1+\cos 6x}{2} + \frac{1+\cos 2x}{2} - \cos 2x - \cos 4x\right) dx$

$= \frac{1}{16}\int \left(1 - \frac{1}{2}\cos 2x - \cos 4x + \frac{1}{2}\cos 6x\right) dx = \frac{1}{16}x - \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{192}\sin 6x + C$

25. $x = a\sin t$, $dx = a\cos t dt$

$\int \sqrt{a^2 - x^2} dx = a^2 \int \cos^2 t dt = \frac{a^2}{2}\int (1+\cos 2t) dt = \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2} + C$

$$\int \sqrt{1-4x^2}\,dx = \frac{1}{4}\sin^{-1}(2x) + \frac{1}{2}x\sqrt{1-4x^2} + C$$

26. $x = 2\tan t$, $dx = 2\sec^2 t dt$

$$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{\sec t}{4\tan^2 t}dt = \frac{1}{4}\int \csc t \cot t dt = -\frac{1}{4}\csc t + C = -\frac{\sqrt{x^2+4}}{4x} + C$$

27. $x+1 = \tan t$, $dx = \sec^2 t dt$

$$\int \frac{1}{(x+1)\sqrt{x^2+2x+2}}dx = \int \frac{\sec^2 t}{\tan t \sec t}dt = \int \csc t dt = \ln|\csc t - \cot t| + C$$

$$= \ln\left|\frac{\sqrt{x^2+2x+2}-1}{x+1}\right| + C$$

28. $$\int \frac{x^2+2x-2}{x^3-4x}dx = \int \left(\frac{1}{2x} + \frac{3}{4(x-2)} - \frac{1}{4(x+2)}\right)dx$$

$$= \frac{1}{2}\ln|x| + \frac{3}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

29. $$\int \frac{1}{x^4-1}dx = \frac{1}{2}\int \left(\frac{1}{x^2-1} - \frac{1}{x^2+1}\right)dx = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$$

30. $$\int (x^2+2x+2)^5(x+1)dx = \frac{1}{12}(x^2+2x+2)^6 + C$$

31. $$\int (x^2-1)\sqrt{x^3-3x}\,dx = \frac{2}{9}(x^3-3x)^{\frac{3}{2}} + C$$

32. $$\int (1+\sin x)^2\cos x dx = \frac{1}{3}(1+\sin x)^3 + C$$

33. $$\int \sin x \cos 2x dx = \frac{1}{2}\int (\sin 3x - \sin x)dx = \frac{1}{2}\cos x - \frac{1}{6}\cos 3x + C$$

34. $$\int \frac{e^x}{\sqrt{e^x+1}}dx = 2\sqrt{e^x+1} + C$$

35. $\int \frac{2x+1}{x^2+x+1}dx = \ln(x^2+x+1)+C$

36. $\int \frac{x+1}{x^3-1}dx = \int \left(\frac{2}{3(x-1)} - \frac{2x+1}{3(x^2+x+1)}\right)dx = \frac{2}{3}\ln|x-1| - \frac{1}{3}\ln(x^2+x+1)+C$

37. $\int \tan 2x dx = \frac{1}{2}\ln|\sec 2x| + C$

38. $\int \frac{1}{x\ln x}dx = \ln|\ln x| + C$

39. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \ln(e^x + e^{-x}) + C$

40. $t = \sqrt{1+2x}$, $t^2 = 1+2x$, $tdt = dx$

$\int (2+3x)\sqrt{1+2x}\,dx = \int t\left(2t + \frac{3}{2}t(t^2-1)\right)dt = \int \left(\frac{1}{2}t^2 + \frac{3}{2}t^4\right)dt = \frac{3}{10}t^5 + \frac{1}{6}t^3 + C$

$= \frac{3}{10}(1+2x)^{\frac{5}{2}} + \frac{1}{6}(1+2x)^{\frac{3}{2}} + C$

41. $\int \frac{x-1}{\sqrt{x+1}}dx = \int \left(\sqrt{x+1} - \frac{2}{\sqrt{x+1}}\right)dx = -4(x+1)^{\frac{1}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

42. $t = 1-x^2$, $dt = -2xdx$

$\int \frac{x^3}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{1-t}{\sqrt{t}}dt = -\frac{1}{2}\int (t^{-\frac{1}{2}} - t^{\frac{1}{2}})dt = -t^{\frac{1}{2}} + \frac{1}{3}t^{\frac{3}{2}} + C = -(1-x^2)^{\frac{1}{2}} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$

43. $\int \frac{1}{1+\cos^2 x}dx = \int \sec^2 x \cdot \frac{1}{\tan^2 x + 2}dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$

44. $\int \frac{x}{(2x+1)^3}dx = \frac{1}{2}\int \left(\frac{1}{(2x+1)^2} - \frac{1}{(2x+1)^3}\right)dx = -\frac{1}{4(2x+1)} + \frac{1}{8(2x+1)^2} + C$

45. $\int \frac{1}{x^3(x+1)}dx = \int \left(\frac{1}{x^3} - \frac{1}{x^2(x+1)}\right)dx = \int \left(\frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - \frac{1}{x+1}\right)dx$

$= \ln\left|\frac{x}{x+1}\right| + \frac{1}{x} - \frac{1}{2x^2} + C$

46. sol 1) $\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos^2 x}{\sin^2 x}dx = \int (\csc^2 x - \csc x \cot x)dx = \csc x - \cot x + C$

sol 2) $\int \frac{1}{1+\cos x}dx = \int \frac{1}{2\cos^2 \frac{x}{2}}dx = \frac{1}{2}\int \sec^2 \frac{x}{2}dx = \tan\frac{x}{2} + C$

47. $\int \frac{x}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + C$

48. $\int \frac{e^x - 1}{e^x + 1}dx = \int \left(\frac{2e^x}{e^x+1} - 1\right)dx = 2\ln(e^x + 1) - x + C$

49. $\int \frac{1}{e^{2x}+1}dx = \int \left(1 - \frac{e^{2x}}{e^{2x}+1}\right)dx = x - \frac{1}{2}\ln(e^{2x}+1) + C$

50. $\int \frac{1}{e^{2x}+e^x}dx = \int \left(\frac{1}{e^x} - \frac{1}{e^x+1}\right)dx = \int \left(e^{-x} - 1 + \frac{e^x}{e^x+1}\right)dx = \ln(e^x+1) - x - e^{-x} + C$

51. $t = e^x + 1$, $dt = e^x dx$

$\int \frac{e^x \ln(e^x+1)}{e^x+1}dx = \int \frac{\ln t}{t}dt = \frac{1}{2}\{\ln t\}^2 + C = \frac{1}{2}\{\ln(e^x+1)\}^2 + C$

52. $t = e^x + 3$, $dt = e^x dx$

$\int \frac{e^x(e^x+1)}{(e^x+3)^2}dx = \int \frac{t-2}{t^2}dt = \ln|t| + \frac{2}{t} + C = \ln|e^x+3| + \frac{2}{e^x+3} + C$

53. $t = \ln x + 1$, $dt = \frac{1}{x}dx$

$\int \frac{\ln x}{x(\ln x+1)^2}dx = \int \frac{t-1}{t^2}dt = \ln|t| + \frac{1}{t} + C = \ln|\ln x + 1| + \frac{1}{\ln x + 1} + C$

54. $\int \frac{x^4}{(x-1)^3}dx = \int \left(x + 3 + \frac{6}{x-1} + \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}\right)dx$

$= \frac{1}{2}x^2 + 3x + 6\ln|x-1| - \frac{4}{x-1} - \frac{1}{2(x-1)^2} + C$

55. $\int \frac{x^2}{(x^2-a^2)(x^2-b^2)}dx = \frac{1}{a^2-b^2}\int\left(\frac{a^2}{x^2-a^2}-\frac{b^2}{x^2-b^2}\right)dx$

$= \frac{1}{2(a^2-b^2)}\left\{a\ln\left|\frac{x-a}{x+a}\right|-b\ln\left|\frac{x-b}{x+b}\right|\right\}+C$

56. $t=x^2+a$, $dt=2xdx$

$\int x^3\sqrt{x^2+a}\,dx == \frac{1}{2}\int(t-a)\sqrt{t}\,dt = \frac{1}{2}\int(t^{\frac{3}{2}}-at^{\frac{1}{2}})dt = \frac{1}{5}t^{\frac{5}{2}}-\frac{1}{3}at^{\frac{3}{2}}+C$

$= \frac{1}{5}(x^2+a)^{\frac{5}{2}}-\frac{1}{3}a(x^2+a)^{\frac{3}{2}}+C$

57. $t=\sqrt[4]{x}$, $dx=4t^3dt$

$\int\frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx = \int\frac{4t^5}{t^3+1}dt = \int\left(4t^2-\frac{4t^2}{t^3+1}\right)dt = \frac{4}{3}t^3-\frac{4}{3}\ln|t^3+1|+C = \frac{4}{3}\sqrt[4]{x^3}-\frac{4}{3}\ln|\sqrt[4]{x^3}+1|+C$

58. $\int\frac{1}{1+\tan x}dx = \int\frac{\cos x}{\cos x+\sin x}dx = \int\left(\frac{1}{2}+\frac{\cos x-\sin x}{2(\cos x+\sin x)}\right)dx$

$= \frac{1}{2}x+\frac{1}{2}\ln|\cos x+\sin x|+C$

59. $\int\frac{1}{\sin 2x-\sin x}dx = \int\frac{1}{\sin x(2\cos x-1)}dx = \int\frac{-\sin x}{(\cos^2x-1)(2\cos x-1)}dx$

$= \int(-\sin x)\left\{\frac{1}{2(\cos x-1)}+\frac{1}{6(\cos x+1)}-\frac{4}{3(2\cos x-1)}\right\}dx$

$= \frac{1}{2}\ln|\cos x-1|+\frac{1}{6}\ln|\cos x+1|-\frac{2}{3}\ln|2\cos x-1|+C$

60. $\int\frac{1}{\sin x}dx = \int\csc x\,dx = \int\frac{\csc^2x-\csc x\cot x}{\csc x-\cot x}dx = \ln|\csc x-\cot x|+C$

61. $\int\frac{1}{1-3\cos^2x}dx = \int\sec^2x\cdot\frac{1}{\tan^2x-2}dx = \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x-\sqrt{2}}{\tan x+\sqrt{2}}\right|+C$

62. $\int\frac{\sin x}{1+\sin x}dx = \int\frac{\sin x-\sin^2x}{\cos^2x}dx = \int(\tan x\sec x-\tan^2x)dx = \sec x-\tan x+x+C$

63. $t=1+\sqrt{x}$, $x=(t-1)^2$, $dx=2(t-1)dt$

$$\int\sqrt{1+\sqrt{x}}\,dx=2\int(t-1)\sqrt{t}\,dt=2\int(t^{\frac{3}{2}}-t^{\frac{1}{2}})dt=\frac{4}{5}t^{\frac{5}{2}}-\frac{4}{3}t^{\frac{3}{2}}+C$$

$$=\frac{4}{5}(1+\sqrt{x})^{\frac{5}{2}}-\frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}}+C$$

64. $t=x+\sqrt{x^2+3}$, $x=\dfrac{t^2-3}{2t}$, $dx=\dfrac{t^2+3}{2t^2}dt$

$$\int\sqrt{x+\sqrt{x^2+3}}\,dx=\frac{1}{2}\int(t^{\frac{1}{2}}-3t^{-\frac{3}{2}})dt=\frac{1}{3}t^{\frac{3}{2}}-3t^{-\frac{1}{2}}+C$$

$$=\frac{1}{3}(x+\sqrt{x^2+3})^{\frac{3}{2}}-3(x+\sqrt{x^2+3})^{-\frac{1}{2}}+C$$

65. $$\int\frac{x}{\sqrt{5-4x-x^2}}dx=\int\frac{x+2}{\sqrt{5-4x-x^2}}dx-\int\frac{2}{\sqrt{3^2-(x+2)^2}}dx$$

$$=-\sqrt{5-4x-x^2}-2\sin^{-1}\left(\frac{x+2}{3}\right)+C$$

66. $$\int\frac{x^3-x-2}{x^3-x^2+x-1}dx=\int\left(1-\frac{1}{x-1}+\frac{2x}{x^2+1}\right)dx=x-\ln|x-1|+\ln(x^2+1)+C$$

67. $$\int\frac{1}{x^2+1}dx=\tan^{-1}x+C$$

68. $$\int\frac{1}{ax^2+bx+c}dx=\frac{1}{a}\int\frac{1}{\left(x+\dfrac{b}{2a}\right)^2+\dfrac{4ac-b^2}{4a^2}}dx=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C$$

69. $$\int\frac{1}{x^2+x+1}dx=\int\frac{1}{\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}}dx=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

70. $t=\tan\dfrac{x}{2}$, $dt=\dfrac{1}{2}\sec^2\dfrac{x}{2}dx$, $dx=2\cos^2\dfrac{x}{2}dt=\dfrac{2}{t^2+1}dt$

$$\int \frac{1}{3+5\cos x}dx = \int \frac{1}{4-t^2}dt = -\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| + C = -\frac{1}{4}\ln\left|\frac{\tan\frac{x}{2}-2}{\tan\frac{x}{2}+2}\right| + C$$

71. $t=\tan\frac{x}{2}$, $dt=\frac{1}{2}\sec^2\frac{x}{2}dx$, $dx=2\cos^2\frac{x}{2}dt=\frac{2}{t^2+1}dt$

$$\int \frac{1}{4\sin x+3\cos x}dx = \int \frac{2}{-3t^2+8t+3}dt = -\frac{1}{5}\ln\left|\frac{t-3}{t+\frac{1}{3}}\right| + C = -\frac{1}{5}\ln\left|\frac{\tan\frac{x}{2}-3}{\tan\frac{x}{2}+\frac{1}{3}}\right| + C$$

72. $x=a\sin t$, $dx=a\cos t$

$$\int \frac{1}{\sqrt{(a^2-x^2)^3}}dx = \frac{1}{a^2}\int \sec^2 t\,dt = \frac{1}{a^2}\tan t + C = \frac{x}{a^2\sqrt{a^2-x^2}} + C$$

73. $x=a\tan t$, $dx=a\sec^2 t\,dt$

$$\int \frac{1}{\sqrt{(a^2+x^2)^3}}dx = \frac{1}{a^2}\int \cos t\,dt = \frac{1}{a^2}\sin t + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C$$

74. $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

75. sol 1) $\int \sin^3 x\,dx = \int \frac{3\sin x - \sin 3x}{4}dx = -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$

sol 2) $\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx = \int \sin x\,dx - \int \cos^2 x\sin x\,dx$

$= -\cos x + \frac{1}{3}\cos^3 x + C$

76. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x + C$

77. $\int \tan^3 x\,dx = \int \tan x(\sec^2 x - 1)dx = \int \tan x\sec^2 x\,dx - \int \tan x\,dx$

$= \frac{1}{2}\tan^2 x - \ln|\sec x| + C$

78. $\int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$= \sec x \tan x + \int \sec x dx - \int \sec^3 x dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C$

79. $\int \frac{1}{4x^2+4x+2} dx = \int \frac{1}{(2x+1)^2+1} dx = \frac{1}{2} \tan^{-1}(2x+1) + C$

80. $\int \frac{1}{\sqrt{x^2+1}} dx = \ln \left| x + \sqrt{x^2+1} \right| + C$

81. $\int \frac{1}{\sqrt{x^2-4}} dx = \ln \left| x + \sqrt{x^2-4} \right| + C$

82. $\int \frac{1}{\sqrt{x^2-2x+2}} dx = \int \frac{1}{\sqrt{(x-1)^2+1}} dx = \ln \left| x - 1 + \sqrt{x^2-2x+2} \right| + C$

83. $x = 3\sin t$, $dx = 3\cos t dt$

$\int \frac{\sqrt{9-x^2}}{x^2} dx = \int \cot^2 t dt = -\cot t - t + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$

84. $\int \frac{4x}{(x-1)^2(x+1)} dx = \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right) dx = \ln \left| \frac{x-1}{x+1} \right| - \frac{2}{x-1} + C$

85. $\int \frac{2x^2-x+4}{x(x^2+4)} dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx = \ln |x| + \frac{1}{2} \ln(x^2+4) - \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$

86. $\int \frac{1}{16x^4-1} dx = \frac{1}{2} \int \left(\frac{1}{4x^2-1} - \frac{1}{4x^2+1}\right) dx = \frac{1}{8} \ln \left| \frac{2x-1}{2x+1} \right| - \frac{1}{4} \tan^{-1}(2x) + C$

87. $\int \frac{x^2+1}{(x-1)(x-2)(x-3)} dx = \int \left(\frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}\right) dx$

$= \ln|x-1| - 5\ln|x-2| + 5\ln|x-3| + C$

88. $\int \frac{x+4}{x^3+3x^2-10x} dx = \int \frac{x+4}{x(x+5)(x-2)} dx = \int \left(-\frac{2}{5x} - \frac{1}{35(x+5)} + \frac{3}{7(x-2)}\right) dx$

$= -\frac{2}{5} \ln|x| - \frac{1}{35} \ln|x+5| + \frac{3}{7} \ln|x-2| + C$

89. $x=\tan t$, $dx=\sec^2 t dt$

$$\int \sqrt{x^2+1}\,dx=\int \sec^3 t dt=\frac{1}{2}\sec t\tan t+\frac{1}{2}\ln|\sec t+\tan t|+C\ (\because 78)$$

$$=\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$$

90. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}dx=\sin^{-1}(e^x)+C$

91. $t=x^2$, $dt=2xdx$

$$\int \frac{1}{x(1+x^2)}dx=\int \frac{1}{2t(t+1)}dt=\frac{1}{2}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt=\frac{1}{2}\ln\left|\frac{t}{t+1}\right|+C=\frac{1}{2}\ln\left|\frac{x^2}{x^2+1}\right|+C$$

92. $x=\tan t$, $dx=\sec^2 t dt$

$$\int \frac{1}{x^2\sqrt{x^2+1}}dx=\int \frac{1}{\tan^2 t\sec t}\cdot\sec^2 t dt=\int \cot t\csc t dt=-\csc t+C=-\frac{\sqrt{x^2+1}}{x}+C$$

93. $\displaystyle\int \frac{1}{x\sqrt{1-x^2}}dx=\int \frac{x}{x^2\sqrt{1-x^2}}dx=\int \frac{-1}{1-t^2}dt=\frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right|+C$

94. $\displaystyle\int \frac{1}{\sqrt{(x-\alpha)(x-\beta)}}dx=\int \frac{1}{\sqrt{\left(x-\frac{\alpha+\beta}{2}\right)^2-\left(\frac{\alpha-\beta}{2}\right)^2}}dx$

$$=\ln\left|x-\frac{\alpha+\beta}{2}+\sqrt{(x-\alpha)(x-\beta)}\right|+C$$

95. $\displaystyle\int \frac{1}{3\cos^2 x-\sin^2 x}dx=\int \sec^2 x\cdot\frac{1}{3-\tan^2 x}dx=\frac{1}{2\sqrt{3}}\ln\left|\frac{\tan x+\sqrt{3}}{\tan x-\sqrt{3}}\right|+C$

96. $\displaystyle\int \frac{7x^3-13x^2-24x+24}{x^4-3x^3-10x^2+24x}dx=\int \frac{7x^3-13x^2-24x+24}{x(x-2)(x+3)(x-4)}dx$

$$=\int\left(\frac{1}{x-2}+\frac{1}{x}+\frac{2}{x+3}+\frac{3}{x-4}\right)dx=\ln|x-2|+\ln|x|+2\ln|x+3|+3\ln|x-4|+C$$

97. $t=\sqrt[4]{x}$, $dt=\frac{1}{4\sqrt[4]{x^3}}dx$

$$\int\frac{1}{\sqrt{x}+\sqrt[4]{x^3}}dx=\int\frac{1}{t^2+t^3}\cdot 4t^3dt=4\int\frac{t}{t+1}dt=4\int\left(1-\frac{1}{t+1}\right)dt=4t-4\ln|t+1|+C$$

$$=4\sqrt[4]{x}-4\ln|\sqrt[4]{x}+1|+C$$

98. sol 1) $t=\frac{1}{x}$, $dt=-\frac{1}{x^2}dx$

$$I=\int\frac{1}{x^2\sqrt{2x-x^2}}dx=\int\frac{1}{x^3\sqrt{\frac{2}{x}-1}}dx=-\int\frac{t}{\sqrt{2t-1}}dt=-\frac{1}{2}\int\frac{2t-1+1}{\sqrt{2t-1}}dt$$

$$=-\frac{1}{2}\int\left(\sqrt{2t-1}+\frac{1}{\sqrt{2t-1}}\right)dt=-\frac{1}{6}(2t-1)^{\frac{3}{2}}-\frac{1}{2}\sqrt{2t-1}+C=-\frac{\sqrt{2t-1}(t+1)}{3}+C$$

$$=-\frac{\sqrt{2x-x^2}(x+1)}{3x^2}+C$$

sol 2) $\sin t=x-1$, $\cos t\,dt=dx$, $u=\tan\frac{t}{2}$, $du=\frac{1}{2}\sec^2\frac{t}{2}dt$

$$I=\int\frac{1}{x^2\sqrt{2x-x^2}}dx=\int\frac{1}{x^2\sqrt{1-(x-1)^2}}dx=\int\frac{1}{(\sin t+1)^2\cos t}\cos t\,dt$$

$$=\int\frac{1}{(\sin t+1)^2}dt=\int\frac{(u^2+1)^2}{(u+1)^4}\cdot\frac{2}{u^2+1}du=\int\frac{2(u^2+1)}{(u+1)^4}du$$

$$=2\int\left(\frac{1}{(u+1)^2}-\frac{2}{(u+1)^3}+\frac{2}{(u+1)^4}\right)du=-\frac{2}{u+1}+\frac{2}{(u+1)^2}-\frac{4}{3(u+1)^3}+C$$

$$=-\frac{2(3u^2+3u+2)}{3(u+1)^3}+C=-\frac{\sqrt{2x-x^2}(x+1)}{3x^2}+C$$

※ $u=\sqrt{\frac{1-\sqrt{2x-x^2}}{1+\sqrt{2x-x^2}}}$

99. $x=\sec t$, $dx=\tan t\sec t\,dt$

$$\int \frac{1}{x+\sqrt{x^2-1}}dx=\int (x-\sqrt{x^2-1})dx=\frac{1}{2}x^2-\int \sqrt{x^2-1}\,dx=\frac{1}{2}x^2-\int \tan^2 t\sec t dt$$

$$=\frac{1}{2}x^2-\int \sec t(\sec^2 t-1)dt=\frac{1}{2}x^2-\int \sec^3 t dt+\int \sec t dt$$

$$=\frac{1}{2}x^2-\frac{1}{2}\sec t\tan t+\frac{1}{2}\ln|\sec t+\tan t|+C \ (\because\ [1])$$

$$=\frac{1}{2}x^2-\frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\ln\left|x+\sqrt{x^2-1}\right|+C$$

[1] : $\int \sec^3 t dt=\int \sec t\cdot\sec^2 t dt=\sec t\tan t-\int \sec t\tan t\cdot\tan t dt$

$$=\sec t\tan t-\int \sec t(\sec^2 t-1)dt=\sec t\tan t+\ln|\sec t+\tan t|-\int \sec^3 t dt$$

$$=\frac{1}{2}\sec t\tan t+\frac{1}{2}\ln|\sec t+\tan t|+C$$

100. $2\sin t=x-1$, $2\cos t dt=dx$

$$\int \frac{1}{(x-1)^2\sqrt{3+2x-x^2}}dx=\int \frac{1}{(x-1)^2\sqrt{4-(x-1)^2}}dx=\int \frac{1}{4\sin^2 t\cdot 2\cos t}\cdot 2\cos t dt$$

$$=\frac{1}{4}\int \csc^2 t dt=-\frac{1}{4}\cot t+C=\frac{\sqrt{3+2x-x^2}}{4(1-x)}+C$$

101. $t=x^2+\frac{1}{x^2}-1$, $dt=2\left(x-\frac{1}{x^3}\right)dx$

$$\int \frac{x^4-1}{x^2\sqrt{x^4-x^2+1}}dx=\int \frac{x^4-1}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}dx=\int \frac{x-1/x^3}{\sqrt{x^2+\frac{1}{x^2}-1}}dx=\frac{1}{2}\int \frac{1}{\sqrt{t}}dt=\sqrt{t}+C$$

$$=\sqrt{x^2+\frac{1}{x^2}-1}+C=\frac{\sqrt{x^4-x^2+1}}{x}+C$$

102. $t=x^2$, $dt=2xdx$

$$\int \frac{x^4+81}{x(x^2+9)^2}dx=\frac{1}{2}\int \frac{t^2+81}{t(t+9)^2}dt=\frac{1}{2}\int\left(\frac{1}{t}-\frac{18}{(t+9)^2}\right)dt=\frac{1}{2}\ln|t|+\frac{9}{t+9}+C=\ln|x|+\frac{9}{x^2+9}+C$$

103. $\int \frac{x^2-x+2}{x^3-1}dx = \int\left(\frac{x-4}{3(x^2+x+1)}+\frac{2}{3(x-1)}\right)dx = \int\left(\frac{2x+1-9}{6(x^2+x+1)}+\frac{2}{3(x-1)}\right)dx$

$= \frac{1}{6}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{2}\int \frac{1}{x^2+x+1}dx + \frac{2}{3}\int \frac{1}{x-1}dx$

$= \frac{1}{6}\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{2}{3}\ln|x-1| + C$

104. $t=\sqrt{x}$, $dt=\frac{1}{2\sqrt{x}}dx$, $t=2\sin\theta$, $dt=2\cos\theta d\theta$, $\sin\theta=\frac{\sqrt{x}}{2}$

$\int\sqrt{\frac{4-x}{x}}dx = \int \frac{\sqrt{4-t^2}}{t}\cdot 2tdt = 2\int\sqrt{4-t^2}dt = 2\int 4\cos^2\theta d\theta = 4\int(1+\cos 2\theta)d\theta = 4\theta + 4\sin\theta\cos\theta + C$

$= 4\left(\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\frac{\sqrt{x}}{2}\cdot\frac{\sqrt{4-x}}{2}\right)+C = 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right)+\sqrt{x(4-x)}+C$

105. $t=x^{\frac{3}{2}}$, $dt=\frac{3}{2}\sqrt{x}\,dx$

$\int\sqrt{\frac{x}{1-x^3}}dx = \frac{2}{3}\int\frac{1}{\sqrt{1-t^2}}dt = \frac{2}{3}\sin^{-1}t + C = \frac{2}{3}\sin^{-1}x^{\frac{3}{2}} + C$

106. $t=\sqrt{x}$, $dt=\frac{1}{2\sqrt{x}}dx$, $t=\sin u$, $dt=\cos u du$

$\int\sqrt{x}\sqrt{1-x}\,dx = \int t\sqrt{1-t^2}\cdot 2tdt = 2\int t^2\sqrt{1-t^2}dt = 2\int\sin^2 u\cos u\cdot\cos u du = \frac{1}{2}\int\sin^2 2u du$

$= \frac{1}{4}\int(1-\cos 4u)du = \frac{1}{4}u - \frac{1}{16}\sin 4u + C = \frac{1}{4}\left(t\sqrt{1-t^2}(2t^2-1)+\sin^{-1}t\right)+C$

$= \frac{1}{4}(\sqrt{x}\sqrt{1-x}(2x-1)+\sin^{-1}\sqrt{x})+C$

107. $t=\frac{x-2}{x-1}$, $dt=\frac{1}{(x-1)^2}dx$, $x=\frac{t-2}{t-1}$, $t=u^2$, $dt=2udu$

$\int\sqrt{\frac{x-2}{x-1}}dx = \int\frac{\sqrt{t}}{(t-1)^2}dt = \int\frac{u}{(u^2-1)^2}\cdot 2udu = 2\int\frac{u^2}{(u^2-1)^2}du$

$$=\frac{1}{2}\int\left(-\frac{1}{u+1}+\frac{1}{(u+1)^2}+\frac{1}{u-1}+\frac{1}{(u-1)^2}\right)du=\frac{1}{2}\left(-\frac{2u}{u^2-1}+\ln\left|\frac{1-u}{1+u}\right|\right)+C$$

$$=\frac{1}{2}\left(-\frac{2\sqrt{t}}{t-1}+\ln\left|\frac{1-\sqrt{t}}{1+\sqrt{t}}\right|\right)+C=\sqrt{x-1}\sqrt{x-2}+\frac{1}{2}\ln\left|\frac{\sqrt{x-1}-\sqrt{x-2}}{\sqrt{x-1}+\sqrt{x-2}}\right|+C$$

108. $x=5\sec t$, $dx=5\tan t\sec t dt$

$$\int\frac{\sqrt{x^2-25}}{x^3}dx=\int\frac{5\tan t}{125\sec^3 t}\cdot 5\tan t\sec t dt=\frac{1}{5}\int\sin^2 t dt=\frac{1}{10}\int(1-\cos 2t)dt$$

$$=\frac{1}{10}t-\frac{1}{20}\sin 2t+C=\frac{1}{10}\sec^{-1}\left(\frac{x}{5}\right)-\frac{\sqrt{x^2-25}}{2x^2}+C$$

109. $t=\tan x$, $dt=\sec^2 x dx$

$$\int\frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx=\int\frac{\tan x\sec^2 x}{1+\tan^4 x}dx=\int\frac{t}{1+t^4}dt=\frac{1}{2}\tan^{-1}(t^2)+C=\frac{1}{2}\tan^{-1}(\tan^2 x)+C$$

110. $\int\frac{1}{\sin^2 x\cos^2 x}dx=\int\csc^2 x\sec^2 x dx=4\int\csc^2 2x dx=-2\cot 2x+C$

111. $x=\tan t$, $dx=\sec^2 t dt$

$$\int\frac{1}{(1+x^2)^2}dx=\int\frac{1}{\sec^4 t}\cdot\sec^2 t dt=\int\cos^2 t dt=\frac{1}{2}\int(1+\cos 2t)dt=\frac{1}{2}t+\frac{1}{4}\sin 2t+C$$

$$=\frac{1}{2}\left(\tan^{-1}x+\frac{x}{x^2+1}\right)+C$$

112. $\int\tan^6 x\sec^4 x dx=\int\tan^6 x(\tan^2 x+1)\sec^2 x dx=\int\tan^8 x\sec^2 x dx+\int\tan^6 x\sec^2 x dx$

$$=\frac{1}{9}\tan^9 x+\frac{1}{7}\tan^7 x+C$$

113. $\int\tan^4 x\sec^4 x dx=\int\tan^4 x(\tan^2 x+1)\sec^2 x dx=\int\tan^6 x\sec^2 x dx+\int\tan^4 x\sec^2 x dx$

$$=\frac{1}{7}\tan^7 x+\frac{1}{5}\tan^5 x+C$$

114. $\int \tan x \sec^5 x dx = \int \tan x \sec x \sec^4 x dx = \frac{1}{5}\sec^5 x + C$

115. $t = \sec x$, $dt = \tan x \sec x dx$

$\int \tan^5 x \sec^7 x dx = \int \tan x \sec^7 x (\sec^2 x - 1)^2 dx = \int t^6 (t^2 - 1)^2 dt = \int (t^{10} - 2t^8 + t^6) dt$

$= \frac{t^{11}}{11} - \frac{2t^9}{9} + \frac{t^7}{7} + C = \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C$

116. $\int x \sec^2 x dx = x \tan x - \int \tan x dx = x \tan x + \ln|\sec x| + C$

117. $\int x \sin(x+2) dx = -x\cos(x+2) + \int \cos(x+2) dx = -x\cos(x+2) + \sin(x+2) + C$

118. $\int \ln(x^2 - x) dx = \int \{\ln x + \ln(x-1)\} dx = x\ln x - x + (x-1)\ln(x-1) - x + C$

$= x\ln x + (x-1)\ln(x-1) - 2x + C$

119. $\int e^x \sin^2 \frac{x}{2} dx = \int e^x \left(\frac{1-\cos x}{2}\right) dx = \frac{1}{2}e^x - \frac{1}{2}\int e^x \cos x dx = \frac{e^x}{4}(2 - \sin x - \cos x) + C$

120. $\int \frac{2x+7}{\sqrt{x^2+6x-7}} dx = \int \left(\frac{1}{\sqrt{(x+3)^2 - 4^2}} + \frac{2x+6}{\sqrt{x^2+6x+7}}\right) dx$

$= \ln\left|x + 3 + \sqrt{x^2+6x-7}\right| + 2\sqrt{x^2+6x-7} + C$

121. $\int \frac{2x-1}{x^2-2x-3} dx = \int \left(\frac{5}{4(x-3)} + \frac{3}{4(x+1)}\right) dx = \frac{5}{4}\ln|x-3| + \frac{3}{4}\ln|x+1| + C$

122. $\int 4\sin 2x \cos 2x \cos 4x dx = \int 2\sin 4x \cos 4x dx = \int \sin 8x dx = -\frac{1}{8}\cos 8x + C$

123. $\int (2x+3)e^x dx = (2x+3)e^x - 2\int e^x dx = (2x+1)e^x + C$

124. $\int x \ln x dx = x(x\ln x - x) - \int (x\ln x - x) dx$ $\left(\because \int \ln x dx = x\ln x - x + C\right)$

$= x^2 \ln x - x^2 + \frac{1}{2}x^2 - \int x\ln x dx = \frac{1}{4}x^2(2\ln x - 1) + C$

125. $\int x^2e^x dx = x^2e^x - \int 2xe^x dx = x^2e^x - 2xe^x + 2\int e^x dx = e^x(x^2-2x+2)+C$

126. $\int x^2\ln x dx = \frac{1}{3}x^3\ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}dx = \frac{1}{3}x^3\ln x - \frac{1}{3}\int x^2 dx = \frac{1}{9}x^3(3\ln x - 1)+C$

127. $\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + C$

128. $\int (\ln x)^2 dx = x(\ln x)^2 - \int \frac{2\ln x}{x} \cdot x dx = x(\ln x)^2 - 2\int \ln x dx = x\{(\ln x)^2 - 2\ln x + 2\} + C$

129. $\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx = e^x(\sin x - \cos x) - \int e^x \sin x dx$

$= \frac{1}{2}e^x(\sin x - \cos x) + C$

130. $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx = e^x(\sin x + \cos x) - \int e^x \cos x dx$

$= \frac{1}{2}e^x(\sin x + \cos x) + C$

131. $\int e^x \sin^2 x dx = \frac{1}{2}\int e^x(1-\cos 2x)dx = \frac{1}{2}e^x - \frac{1}{5}e^x(\sin 2x + \frac{1}{2}\cos 2x) + C$

$= \frac{1}{10}e^x(5 - 2\sin 2x - \cos 2x) + C$

132. ① $n \neq -1$, $\int x^n \ln x dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n dx$

$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + C = \frac{x^{n+1}}{(n+1)^2}\{(n+1)\ln x - 1\} + C$

② $n = -1$, $\int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2 + C$

133. $\int \ln x dx = x\ln x - x + C$

134. $\int \ln(x+1)dx = (x+1)\ln(x+1) - x + C$

135. $\int \frac{1}{1+e^{x}}dx=\int\left(1-\frac{e^{x}}{1+e^{x}}\right)dx=x-\ln(1+e^{x})+C$

136. $\int \frac{x^3+x^2+1}{x^2(x^2+1)}dx=\int\left(\frac{x}{x^2+1}+\frac{1}{x^2}\right)dx=\frac{1}{2}\ln|x^2+1|-\frac{1}{x}+C$

137. $\int (x^2+1)e^{x}dx=e^{x}+x^2e^{x}-\int 2xe^{x}dx=(x^2+1)e^{x}-2xe^{x}+2\int e^{x}dx$

$=e^{x}(x^2-2x+3)+C$

138. $t=\sqrt{x+1}$, $dt=\frac{1}{2\sqrt{x+1}}dx$

$\int \frac{1}{x\sqrt{x+1}}dx=\int\frac{2}{t^2-1}dt=-\int\left(\frac{1}{1-t}+\frac{1}{1+t}\right)dt=\ln\left|\frac{t-1}{t+1}\right|+C=\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C$

139. $t=\sqrt{x}$, $dt=\frac{1}{2\sqrt{x}}dx$

$\int \frac{x\sqrt{x}-1}{x-\sqrt{x}}dx=\int\frac{t^3-1}{t^2-t}\cdot 2tdt=2\int(t^2+t+1)dt=\frac{2}{3}t^3+t^2+2t+C=\frac{2}{3}x\sqrt{x}+x+2\sqrt{x}+C$

140. $\int \ln(x+\sqrt{x^2+1})dx=x\ln(x+\sqrt{x^2+1})-\int x\cdot\frac{1+\frac{2x}{2\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}dx$

$=x\ln(x+\sqrt{x^2+1})-\int\frac{x}{\sqrt{x^2+1}}dx=x\ln(x+\sqrt{x^2+1})-\sqrt{x^2+1}+C$

141. $\int\left(x^2-\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)dx=\int\left(x^3-x-\frac{1}{x}+\frac{1}{x^3}\right)dx=\frac{1}{4}x^4-\frac{1}{2}x^2-\ln|x|-\frac{1}{2x^2}+C$

142. $t=\sqrt{x}$, $dt=\frac{1}{2\sqrt{x}}dx$

$\int \sin\sqrt{x}\,dx=2\int t\sin t dt=-2t\cos t+2\int\cos t dt=2\sin t-2t\cos t+C=2\sin\sqrt{x}-2\sqrt{x}\cos\sqrt{x}+C$

143. $\int \ln(x^2+1)dx=x\ln(x^2+1)-\int\frac{2x^2}{x^2+1}dx=x\ln(x^2+1)-\int\left(2-\frac{2}{x^2+1}\right)dx$

$= x\{\ln(x^2+1)-2\}+2\tan^{-1}x+C$

144. $t=x^2$, $dt=2xdx$

$$\int \frac{-2x}{\sqrt{1-x^4}}dx=-\int \frac{1}{\sqrt{1-t^2}}dt=-\sin^{-1}t+C=-\sin^{-1}(x^2)+C$$

145. $t=\sqrt{x}$, $dt=\frac{1}{2\sqrt{x}}dx$

$$\int e^{\sqrt{x}}dx=2\int te^t dt=2te^t-2\int e^t dt=2e^t(t-1)+C=2e^{\sqrt{x}}(\sqrt{x}-1)+C$$

146. $x=a\sin t$, $dx=a\cos t dt$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx=\int \frac{a^2\sin^2 t}{a\cos t}\cdot a\cos t dt=a^2\int \sin^2 t dt=\frac{a^2}{2}\int (1-\cos 2t)dt=\frac{a^2}{4}(2t-\sin 2t)+C$$

$$=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)-\frac{x\sqrt{a^2-x^2}}{2}+C$$

147. $x=a\sin t$, $dx=a\cos t dt$

$$\int \frac{1}{x\sqrt{a^2-x^2}}dx=\int \frac{a\cos t}{a\sin t\cdot a\cos t}dt=\frac{1}{a}\int \csc t dt=\frac{1}{a}\ln|\csc t-\cot t|+C$$

$$=\frac{1}{a}\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right|+C$$

148. $t=e^x$, $dt=e^x dx$, $u=\sqrt{t+1}$, $du=\frac{1}{2\sqrt{t+1}}dt$

$$\int \frac{1}{\sqrt{1+e^x}}dx=\int \frac{1}{t\sqrt{t+1}}dt=\int \frac{2}{u^2-1}du=\ln\left|\frac{u-1}{u+1}\right|+C=\ln\left|\frac{\sqrt{t+1}-1}{\sqrt{t+1}+1}\right|+C$$

$$=\ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right|+C$$

149. $t=x^2$, $dt=2xdx$, $t=\tan u$, $dt=\sec^2 u du=(1+t^2)du$

$$\int \frac{x}{\sqrt{1+x^4}}dx=\frac{1}{2}\int \frac{1}{\sqrt{t^2+1}}dt=\frac{1}{2}\int \sec u du=\frac{1}{2}\ln|\tan u+\sec u|+C=\frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right|+C$$

$$=\frac{1}{2}\ln\left|x^2+\sqrt{x^4+1}\right|+C$$

150\. $t=\ln x$, $dt=\frac{1}{x}dx$, $t=\sin u$, $dt=\cos u du$

$$\int \frac{\sqrt{1-(\ln x)^2}}{x\ln x}dx=\int \frac{\sqrt{1-t^2}}{t}dt=\int \frac{\cos^2 u}{\sin u}du=\int \frac{1-\sin^2 u}{\sin u}du=\int (\csc u-\sin u)du$$

$$=\ln|\csc u-\cot u|+\cos u+C=\ln\left|\frac{1-\sqrt{1-t^2}}{t}\right|+\sqrt{1-t^2}+C=\ln\left|\frac{1-\sqrt{1-(\ln x)^2}}{\ln x}\right|+\sqrt{1-(\ln x)^2}+C$$

151\. $\int \frac{x+2}{(x^2+1)(x-1)^3}dx=\int\left(\frac{3-x}{4(x^2+1)}+\frac{1}{4(x-1)}-\frac{1}{(x-1)^2}+\frac{3}{2(x-1)^3}\right)dx$

$$=-\frac{1}{8}\int \frac{2x}{x^2+1}dx+\frac{3}{4}\int \frac{1}{x^2+1}dx+\frac{1}{4}\int \frac{1}{x-1}dx-\int \frac{1}{(x-1)^2}dx+\frac{3}{2}\int \frac{1}{(x-1)^3}dx$$

$$=-\frac{1}{8}\ln|x^2+1|+\frac{3}{4}\tan^{-1}x+\frac{1}{4}\ln|x-1|+\frac{1}{x-1}-\frac{3}{4(x-1)^2}+C$$

152\. $x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan t$, $dx=\frac{\sqrt{3}}{2}\sec^2 t dt$

$$\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx=2\int \frac{\sec t}{\sqrt{3}\tan t+1}dt=2\int \frac{1}{\sqrt{3}\sin t+\cos t}dt$$

$$=\int \sec\left(t-\frac{\pi}{3}\right)dt=\ln\left|\sec\left(t-\frac{\pi}{3}\right)+\tan\left(t-\frac{\pi}{3}\right)\right|+C$$

$$=\ln\left|\frac{\sin t-\sqrt{3}\cos t+2}{\sqrt{3}\sin t+\cos t}\right|+C=\ln\left|\frac{x-1+2\sqrt{x^2+x+1}}{\sqrt{3}(x+1)}\right|+C$$

153\. $x=\tan t$, $dx=\sec^2 t dt=(1+x^2)dt$

$$\int \frac{\tan^{-1}x}{1+x^2}dx=\int t dt=\frac{1}{2}t^2+C=\frac{1}{2}\left(\tan^{-1}x\right)^2+C$$

154\. $x=2\sin t$, $dx=2\cos t dt$

$$\int \frac{4}{x^2\sqrt{4-x^2}}dx = 4\int \frac{1}{4\sin^2 t}dt = \int \csc^2 t dt = -\cot t + C = -\frac{\sqrt{4-x^2}}{x} + C$$

155. $t = 3x$, $dt = 3dx$

$$\int 3\sec^4 3x dx = \int \sec^4 t dt = \int \sec^2 t(1+\tan^2 t)dt = \int \sec^2 t dt + \int \sec^2 t \tan^2 t dt$$

$$= \tan t + \frac{1}{3}\tan^3 t + C = \tan 3x + \frac{1}{3}\tan^3 3x + C$$

156. $\int \cot^3 x dx = \int \cot x(\csc^2 x - 1)dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

157. $\int \csc^4 x dx = \int \csc^2 x(1+\cot^2 x)dx = \int \csc^2 x dx - \int \cot^2 x \cdot (-\csc^2 x)dx$

$$= -\cot x - \frac{1}{3}\cot^3 x + C$$

158. $t = 3x$, $dt = 3dx$

$$\int \tan^2 3x dx = \frac{1}{3}\int \tan^2 t dt = \frac{1}{3}\int (\sec^2 t - 1)dt = \frac{1}{3}\tan t - \frac{1}{3}t + C = \frac{1}{3}\tan 3x - x + C$$

159. $\int \frac{3x^4 - 3x^3 - x^2 - 17x - 2}{(x-3)(x^2+1)^2}dx = \int \left(\frac{2x+3}{x^2+1} + \frac{4x-2}{(x^2+1)^2} + \frac{1}{x-3}\right)dx$

$$= \int \frac{2x}{x^2+1}dx + 3\int \frac{1}{x^2+1}dx + 2\int \frac{2x-1}{(x^2+1)^2}dx + \int \frac{1}{x-3}dx$$

$$= \ln|x^2+1| + 3\tan^{-1}x + 2\left(-\frac{1}{2}\tan^{-1}x + \frac{x^2-x-1}{2(x^2+1)}\right) + \ln|x-3| + C$$

$$= \ln|(x^2+1)(x-3)| + 2\tan^{-1}x + \frac{x^2-x-1}{x^2+1} + C$$

$$= \ln|(x^2+1)(x-3)| + 2\tan^{-1}x - \frac{x+2}{x^2+1} + C$$

※ $x = \tan t$, $dx = \sec^2 t dt$, $\int \frac{2x-1}{(x^2+1)^2}dx = \int \frac{2\tan t - 1}{\sec^4 t} \cdot \sec^2 t dt = \int \frac{2\tan t - 1}{\sec^2 t}dt$

$$=\int(2\sin t\cos t-\cos^2 t)dt=\int\left(\sin 2t-\frac{1+\cos 2t}{2}\right)dt$$

$$=-\frac{1}{2}\cos 2t-\frac{1}{2}t-\frac{1}{4}\sin 2t+C=-\frac{1}{2}\tan^{-1}x+\frac{x^2-x-1}{2(x^2+1)}+C$$

160. $\int\frac{x^2+2x-1}{2x^3+3x^2-2x}dx=\int\left(-\frac{1}{10(x+2)}+\frac{1}{5(2x-1)}+\frac{1}{2x}\right)dx$

$$=\frac{1}{5}\int\frac{1}{2x-1}dx-\frac{1}{10}\int\frac{1}{x+2}dx+\frac{1}{2}\int\frac{1}{x}dx=\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+\frac{1}{2}\ln|x|+C$$

161. $x=\sin t$, $dx=\cos t dt$

$$\int\frac{1}{x-\sqrt{1-x^2}}dx=\int\frac{\cos t}{\sin t-\cos t}dt=\frac{1}{2}\int\left(\frac{\cos t+\sin t}{\sin t-\cos t}-1\right)dt=\frac{1}{2}\ln|\cos t-\sin t|-\frac{1}{2}t+C$$

$$=\frac{1}{2}\ln\left|\sqrt{1-x^2}-x\right|-\frac{1}{2}\sin^{-1}x+C$$

162. $\int\frac{x^2+3}{x^2-1}dx=\int\left(1+\frac{2}{x-1}-\frac{2}{x+1}\right)dx=x+2\ln\left|\frac{x-1}{x+1}\right|+C$

163. $I=\int 13e^{2x}\cos 3x dx=\frac{13}{2}e^{2x}\cos 3x+\frac{3}{2}\int 13e^{2x}\sin 3x dx$

$$=\frac{13}{2}e^{2x}\cos 3x+\frac{39}{2}\left\{\frac{1}{2}e^{2x}\sin 3x-\frac{1}{2}\int 3e^{2x}\cos 3x dx\right\}=\frac{13}{4}e^{2x}(2\cos 3x+3\sin 3x)-\frac{9}{4}\int 13e^{2x}\cos 3x dx$$

$$=\frac{13}{4}e^{2x}(2\cos 3x+3\sin 3x)-\frac{9}{4}I$$

$$\therefore I=e^{2x}(2\cos 3x+3\sin 3x)+C$$

164. $\int\frac{x^4-2x^2+4x+1}{x^3+4x}dx=\int\left(\frac{16-25x}{4(x^2+4)}+x+\frac{1}{4x}\right)dx$

$$=-\frac{25}{8}\int\frac{2x}{x^2+4}dx+4\int\frac{1}{x^2+4}dx+\int x dx+\frac{1}{4}\int\frac{1}{x}dx$$

$$=-\frac{25}{8}\ln|x^2+4|+2\tan^{-1}\left(\frac{x}{2}\right)+\frac{1}{2}x^2+\frac{1}{4}\ln|x|+C$$

165. $\int\frac{2x^2-x+4}{x^3+4x}dx=\int\left(\frac{x-1}{x^2+4}+\frac{1}{x}\right)dx=\frac{1}{2}\int\frac{2x}{x^2+4}dx-\int\frac{1}{x^2+4}dx+\int\frac{1}{x}dx$

$$=\frac{1}{2}\ln|x^2+4|-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\ln|x|+C$$

166. $\int\frac{1}{(x+1)(x^2+1)}dx=\int\left(\frac{1-x}{2(x^2+1)}+\frac{1}{2(x+1)}\right)dx$

$$=\frac{1}{2}\int\frac{1}{x^2+1}dx-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x+1}dx=\frac{1}{2}\tan^{-1}x-\frac{1}{4}\ln|x^2+1|+\frac{1}{2}\ln|x+1|+C$$

167. $\int\frac{x^2-2x-2}{x^3-1}dx=\int\left(\frac{2x+1}{x^2+x+1}-\frac{1}{x-1}\right)dx=\ln|x^2+x+1|-\ln|x-1|+C$

168. $x=a\sin t$, $dx=a\cos t\,dt$

$$\int\sqrt{a^2-x^2}\,dx=a^2\int\cos^2t\,dt=\frac{a^2}{2}\int(1+\cos2t)dt=\frac{a^2}{2}\left(t+\frac{1}{2}\sin2t\right)+C=\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

169. $t=\tan x$, $\sec^2x\,dx=dt$

$$\int\frac{1}{a^2\sin^2x+b^2\cos^2x}dx=\int\frac{\sec^2x}{a^2\tan^2x+b^2}dx=\int\frac{1}{a^2t^2+b^2}dt=\frac{1}{a^2}\int\frac{1}{t^2+\frac{b^2}{a^2}}dt=\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}t\right)+C$$

$$=\frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\tan x\right)+C$$

170. $\int\frac{1}{\sin(x-a)\sin(x-b)}dx=\frac{1}{\sin(b-a)}\int\frac{\sin\{(x-a)-(x-b)\}}{\sin(x-a)\sin(x-b)}dx$

$$=\frac{1}{\sin(b-a)}\int\left(\frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\sin(x-a)\sin(x-b)}\right)dx=\frac{1}{\sin(b-a)}\int(\cot(x-b)-\cot(x-a))dx$$

$$=\frac{\ln|\sin(x-b)|-\ln|\sin(x-a)|}{\sin(b-a)}+C$$

171. $\int x\sec^2x\tan x\,dx=\frac{1}{2}x\sec^2x-\frac{1}{2}\int\sec^2x\,dx=\frac{1}{2}x\sec^2x-\frac{1}{2}\tan x+C$

172. $\int e^{-x}\sin^2 2x dx = \frac{1}{2}\int e^{-x}(1-\cos 4x)dx = -\frac{1}{2}e^{-x} - \frac{1}{2}\int e^{-x}\cos 4x dx \ \cdots \ [1]$

$= -\frac{1}{2}e^{-x} - \frac{1}{2}\left\{-e^{-x}\cos 4x - 4\int e^{-x}\sin 4x dx\right\} = -\frac{1}{2}e^{-x}(1-\cos 4x) + 2\int e^{-x}\sin 4x dx$

$= -\frac{1}{2}e^{-x}(1-\cos 4x) + 2\left\{-e^{-x}\sin 4x + 4\int e^{-x}\cos 4x dx\right\}$

$= -\frac{1}{2}e^{-x}(1-\cos 4x + 4\sin 4x) + 8\int e^{-x}\cos 4x dx \ \cdots \ [2]$

[1], [2]에서

$\int e^{-x}\cos 4x dx = -\frac{1}{17}e^{-x}(\cos 4x - 4\sin 4x) + C$

$\therefore \int e^{-x}\sin^2 2x dx = -\frac{1}{2}e^{-x} - \frac{1}{2}\int e^{-x}\cos 4x dx = \frac{1}{34}e^{-x}(\cos 4x - 4\sin 4x - 17) + C$

173. sol 1) $t^2 = \tan x$, $2tdt = \sec^2 x dx$, $dx = \frac{2t}{t^4+1}dt$

$$\int \sqrt{\tan x}\, dx = \int \frac{2t^2}{t^4+1}dt = \int \frac{2}{t^2+\frac{1}{t^2}}dt = \int \frac{\left(1+\frac{1}{t^2}\right)+\left(1-\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt + \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$$

$$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt + \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt + \int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-(\sqrt{2})^2}dt$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{t+1/t-\sqrt{2}}{t+1/t+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{\tan x}+\sqrt{\cot x}-\sqrt{2}}{\sqrt{\tan x}+\sqrt{\cot x}+\sqrt{2}}\right| + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x-\cos x}{\sqrt{\sin 2x}}\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sin x+\cos x-\sqrt{\sin 2x}}{\sin x+\cos x+\sqrt{\sin 2x}}\right| + C$$

sol 2) $I=\int\sqrt{\tan x}\,dx$, $J=\int\sqrt{\cot x}\,dx$

$$I+J=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx=\sqrt{2}\int\frac{(\sin x-\cos x)'}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

$$=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C_1 \quad\cdots\quad [1]$$

$$I-J=\int(\sqrt{\tan x}-\sqrt{\cot x})dx=\sqrt{2}\int\frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx=-\sqrt{2}\int\frac{(\sin x+\cos x)'}{\sqrt{(\sin x+\cos x)^2-1}}dx$$

$$=-\sqrt{2}\ln\left|(\sin x+\cos x)+\sqrt{(\sin x+\cos x)^2-1}\right|+C_2 \quad\cdots\quad [2]$$

$$[1]+[2]\ :\ I=\int\sqrt{\tan x}\,dx=\frac{1}{\sqrt{2}}\sin^{-1}(\sin x-\cos x)-\frac{1}{\sqrt{2}}\ln\left|\sin x+\cos x+\sqrt{\sin 2x}\right|+C$$

174. $\int\sqrt[3]{\tan x}\,dx$

[1] $t^3=\tan x$, $x=\tan^{-1}(t^3)$, $dx=\dfrac{3t^2}{1+t^6}dt$

[2] $a=t^2$, $da=2tdt$, $a=\tan^{2/3}x$

$$\int\sqrt[3]{\tan x}\,dx=\int\frac{3t^3}{1+t^6}dt \quad\cdots\quad [1]$$

$$=\frac{3}{2}\int\frac{a}{1+a^3}da \quad\cdots\quad [2]$$

$$=\frac{3}{2}\int\left(-\frac{1}{3(a+1)}+\frac{a+1}{3(a^2-a+1)}\right)da=\frac{1}{2}\int\left(\frac{2a-1}{2(a^2-a+1)}+\frac{3}{2(a^2-a+1)}-\frac{1}{a+1}\right)da$$

$$=\frac{1}{4}\ln\frac{a^2-a+1}{(a+1)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(a-\frac{1}{2}\right)\right)+C$$

$$=\frac{1}{4}\ln\frac{\tan^{4/3}x-\tan^{2/3}x+1}{\left(\tan^{2/3}x+1\right)^2}+\frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(\tan^{2/3}x-\frac{1}{2}\right)\right)+C$$

175. $\int\left(\dfrac{x+3}{\sqrt{4-x^2}}+\cot x[\ln(\sin x)]\right)dx=\int\left(\dfrac{x}{\sqrt{4-x^2}}+\dfrac{3}{\sqrt{4-x^2}}+\dfrac{\cos x}{\sin x}[\ln(\sin x)]\right)dx$

$$=-\sqrt{4-x^2}+3\sin^{-1}\left(\frac{x}{2}\right)+\frac{1}{2}[\ln(\sin x)]^2+C$$

176. $\int\left(\frac{4x^2}{x^2+9}+\tan^2 2x\right)dx=\int\left(3-\frac{36}{x^2+9}+\sec^2 2x\right)dx=3x-12\tan^{-1}\left(\frac{x}{3}\right)+\frac{1}{2}\tan 2x+C$

177. $\int\frac{1}{(1+x)(2+x)^2(3+x)^3}dx$

$$=\int\left(\frac{2}{x+2}-\frac{17}{8(x+3)}-\frac{1}{(x+2)^2}-\frac{5}{4(x+3)^2}-\frac{1}{2(x+3)^3}+\frac{1}{8(x+1)}\right)dx$$

$$=2\ln|x+2|-\frac{17}{8}\ln|x+3|+\frac{1}{x+2}+\frac{5}{4(x+3)}+\frac{1}{4(x+3)^2}+\frac{1}{8}\ln|x+1|+C$$

178. $\cot x=t^2$, $-\csc^2 x dx=2tdt$, $dx=-\frac{2t}{1+t^4}dt$

$$\int\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx=\int\frac{1}{1+\sqrt{\cot x}}dx=\int\frac{1}{1+t}\cdot\frac{-2t}{1+t^4}dt=-\int\frac{2t}{(1+t)(1+t^4)}dt$$

$$=-\int\left(\frac{t}{1+t^4}+\frac{t^3}{1+t^4}+\frac{1-t^2}{1+t^4}-\frac{1}{1+t}\right)dt$$

$$=-\int\left(\frac{t}{1+t^4}+\frac{t^3}{1+t^4}-\frac{1-t^{-2}}{(t+t^{-1})^2-2}-\frac{1}{1+t}\right)dt$$

$$=-\int\left(\frac{t}{1+t^4}+\frac{t^3}{1+t^4}-\frac{(t+t^{-1})'}{(t+t^{-1})^2-2}-\frac{1}{1+t}\right)dt$$

$$=-\frac{1}{2}\tan^{-1}(t^2)-\frac{1}{4}\ln|1+t^4|+\frac{1}{2\sqrt{2}}\ln\left|\frac{t+t^{-1}-\sqrt{2}}{t+t^{-1}+\sqrt{2}}\right|+\ln|1+t|+C$$

$$=-\frac{1}{2}\tan^{-1}(\cot x)-\frac{1}{4}\ln|1+\cot^2 x|+\frac{1}{2\sqrt{2}}\ln\left|\frac{\cot x-\sqrt{2\cot x}+1}{\cot x+\sqrt{2\cot x}+1}\right|+\ln|1+\sqrt{\cot x}|+C$$

179. $\tan x=t$, $\sec^2 x dx=dt$, $dt=\frac{1}{1+t^2}dt$

$$\int\frac{\tan x+\tan^3 x}{1+\tan^3 x}dx=\int\frac{t+t^3}{1+t^3}\cdot\frac{1}{1+t^2}dt=\int\frac{t}{1+t^3}dt=\frac{1}{2}\int\frac{t+1+t-1}{t^3+1}dt$$

$$= \frac{1}{2}\int \frac{t+1}{(t+1)(t^2-t+1)}dt - \frac{1}{2}\int \frac{t^2-t+1-t^2}{t^3+1}dt = \frac{1}{2}\int \frac{1}{t^2-t+1}dt - \frac{1}{2}\int \frac{1}{t+1}dt + \frac{1}{6}\int \frac{3t^2}{t^3+1}dt$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) - \frac{1}{2}\ln|t+1| + \frac{1}{6}\ln|t^3+1| + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right) - \frac{1}{2}\ln|\tan x+1| + \frac{1}{6}\ln|\tan^3 x+1| + C$$

180. $t = \tan x$, $dt = \sec^2 x dx$

$$\int \frac{1}{\cos 2x+3}dx = \int \frac{1}{\frac{1-\tan^2 x}{1+\tan^2 x}+3}dx = \int \frac{1+\tan^2 x}{2\tan^2 x+4}dx = \frac{1}{2}\int \frac{\sec^2 x}{\tan^2 x+2}dx = \frac{1}{2}\int \frac{1}{t^2+2}dt$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

181. $\int e^x \tan x(1-2\sec^2 x)dx = \int e^x \tan x dx - \int e^x \cdot 2\tan x \sec^2 x dx$

$$= \int e^x \tan x dx - \int e^x \cdot (\sec^2 x)' dx = \left\{e^x \tan x - \int e^x \sec^2 x dx\right\} - \left\{e^x \sec^2 x - \int e^x \sec^2 x dx\right\}$$

$$= e^x(\tan x - \sec^2 x) + C$$

182. $t = \sqrt{\tan x+1}$, $dt = \frac{\sec^2 x}{2\sqrt{\tan x+1}}dx = \frac{(t^2-1)^2+1}{2t}dx$

$$\int \sqrt{\tan x+1}\,dx = \int \frac{2t^2}{(t^2-1)^2+1}dt = 2\int \frac{t^2}{t^4-2t^2+2}dt = 2\int \frac{1}{t^2+\frac{2}{t^2}-2}dt$$

$$= \int \frac{1+\sqrt{2}/t}{t^2+\frac{2}{t^2}-2}dt + \int \frac{1-\sqrt{2}/t}{t^2+\frac{2}{t^2}-2}dt$$

$$= \int \frac{d\left(t-\frac{\sqrt{2}}{t}\right)}{(t-\sqrt{2}/t)^2+2(\sqrt{2}-1)} + \int \frac{d\left(t+\frac{\sqrt{2}}{t}\right)}{(t+\sqrt{2}/t)^2-2(\sqrt{2}+1)}$$

$$= \frac{1}{\sqrt{2(\sqrt{2}-1)}}\tan^{-1}\left(\frac{t-\sqrt{2}/t}{\sqrt{2(\sqrt{2}-1)}}\right) + \frac{1}{2\sqrt{2(\sqrt{2}+1)}}\ln\left|\frac{t+\sqrt{2}/t-\sqrt{2(\sqrt{2}+1)}}{t+\sqrt{2}/t+\sqrt{2(\sqrt{2}+1)}}\right| + C$$

$$=\frac{1}{\sqrt{2(\sqrt{2}-1)}}\tan^{-1}\left(\frac{\tan x+1-\sqrt{2}}{\sqrt{2(\sqrt{2}-1)(\tan x+1)}}\right)+\frac{1}{2\sqrt{2(\sqrt{2}+1)}}\ln\left|\frac{\tan x+1-\sqrt{2(\sqrt{2}+1)(\tan x+1)}+\sqrt{2}}{\tan x+1+\sqrt{2(\sqrt{2}+1)(\tan x+1)}+\sqrt{2}}\right|+C$$

183. $t=\sqrt{\tan x+2}$, $dt=\frac{\sec^2 x}{2\sqrt{\tan x+2}}dx=\frac{(t^2-2)^2+1}{2t}dx$

$$I=\int\sqrt{\tan x+2}\,dx=\int\frac{2t^2}{(t^2-2)^2+1}dt=2\int\frac{t^2}{t^4-4t^2+5}dt$$

$$=2\int\frac{t^2}{(t^2+\sqrt{4+\sqrt{20}}\,t+\sqrt{5})(t^2-\sqrt{4+\sqrt{20}}\,t+\sqrt{5})}dt$$

$$=2\int\left(\frac{At}{t^2+\sqrt{4+\sqrt{20}}\,t+\sqrt{5}}-\frac{Bt}{t^2-\sqrt{4+\sqrt{20}}\,t+\sqrt{5}}\right)dt,$$

$A=-B=-\frac{1}{4}\sqrt{4+\sqrt{20}}(-2+\sqrt{5})$.

한편 $\int\frac{t}{t^2+bt+c}dt=\frac{1}{2}\ln|t^2+bt+c|-\frac{b}{\sqrt{4c-b^2}}\tan^{-1}\left(\frac{2t+b}{\sqrt{4c-b^2}}\right)+C$, $4c-b^2>0$이고

이때 $4c-b^2=4\sqrt{5}-(4+\sqrt{20})=2\sqrt{5}-4>0$이므로

$$\int\frac{t}{t^2+\sqrt{4+\sqrt{20}}\,t+\sqrt{5}}dt=\frac{1}{2}\ln|t^2+\sqrt{4+\sqrt{20}}\,t+\sqrt{5}|-\frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\tan^{-1}\left(\frac{2t+\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\right)+C$$

$$\int\frac{t}{t^2-\sqrt{4+\sqrt{20}}\,t+\sqrt{5}}dt=\frac{1}{2}\ln|t^2-\sqrt{4+\sqrt{20}}\,t+\sqrt{5}|+\frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\tan^{-1}\left(\frac{2t-\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\right)+C$$

이다. 따라서

$$I=2A\left\{\frac{1}{2}\ln|t^2+\sqrt{4+\sqrt{20}}\,t+\sqrt{5}|-\frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\tan^{-1}\left(\frac{2t+\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\right)\right\}$$

$$-2A\left\{\frac{1}{2}\ln|t^2-\sqrt{4+\sqrt{20}}\,t+\sqrt{5}|+\frac{\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\tan^{-1}\left(\frac{2t-\sqrt{4+\sqrt{20}}}{\sqrt{2\sqrt{5}-4}}\right)\right\}+C.$$

$t=\sqrt{\tan x+2}$를 다시 대입하여 정리하면

$$\therefore I=\frac{(2-\sqrt{5})\sqrt{4+\sqrt{20}}}{4}\ln\left|\frac{\tan x+2+\sqrt{4+\sqrt{20}}\sqrt{\tan x+2}+\sqrt{5}}{\tan x+2-\sqrt{4+\sqrt{20}}\sqrt{\tan x+2}+\sqrt{5}}\right|$$

$$+\frac{\sqrt{4+\sqrt{20}}}{2}\left\{\tan^{-1}\left(\frac{2\sqrt{\tan x+2}-\sqrt{4+\sqrt{20}}}{\sqrt{\sqrt{20}-4}}\right)+\tan^{-1}\left(\frac{2\sqrt{\tan x+2}+\sqrt{4+\sqrt{20}}}{\sqrt{\sqrt{20}-4}}\right)\right\}+C$$

184. $x=\tan t,\ dx=\sec^2 t dt$

$$\int\frac{\tan^{-1}x}{x\sqrt{x^2+1}}\cdot\exp\left(-\frac{\tan^{-1}x}{x}\right)dx=\int\frac{t}{\tan t\sec t}\cdot\exp\left(-\frac{t}{\tan t}\right)\sec^2 t dt=\int\frac{t}{\sin t}e^{-t\cot t}dt$$

$=e^{-t\cot t}\cdot u(t)+C$라 하면

$t\csc t e^{-t\cot t}=(e^{-t\cot t}\cdot u(t))'=(t\csc^2 t-\cot t)e^{-t\cot t}\cdot u(t)+e^{-t\cot t}\cdot u'(t)$

$t\csc t=(t\csc^2 t-\cot t)u(t)+u'(t)$

$t\sin t=(t-\cos t\sin t)u(t)+u'(t)\sin^2 t$

에서 $u(t)=\sin t$가 근이 된다.

$$e^{-t\cot t}\cdot u(t)+C=e^{-t\cot t}\cdot\sin t+C=\exp\left(-\frac{\tan^{-1}x}{x}\right)\cdot\frac{x}{\sqrt{1+x^2}}+C$$

185. $x=\cot t,\ \sin(\cot^{-1}x)=\sin t=\dfrac{x}{\sqrt{1+x^2}},\ x=\tan u,\ \cos(\tan^{-1}x)=\cos u=\dfrac{1}{\sqrt{1+x^2}}$

$$\int\cos^2(\tan^{-1}(\sin(\cot^{-1}x)))dx=\int\cos^2\left(\tan^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)dx=\int\frac{1+x^2}{2+x^2}dx$$

$$=\int\left(1-\frac{1}{2+x^2}\right)dx=x-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$$

186. $t=\sin x,\ dt=\cos x dx,\ u=\dfrac{1}{t},\ dt=-\dfrac{1}{u^2}du,\ w=\sqrt{1-u},\ dw=-\dfrac{1}{2\sqrt{1-u}}du$

$$\int\frac{\sec x-\tan x}{\sqrt{\sin^2 x-\sin x}}dx=\int\frac{1-\sin x}{\cos x\sqrt{\sin^2 x-\sin x}}dx=\int\frac{(1-\sin x)\cos x}{(1-\sin^2 x)\sqrt{\sin^2 x-\sin x}}dx$$

$$=\int\frac{\cos x}{(1+\sin x)\sqrt{\sin^2x-\sin x}}dx=\int\frac{1}{(1+t)\sqrt{t^2-t}}dt=-\int\frac{1}{(u+1)\sqrt{1-u}}du$$

$$=2\int\frac{1}{2-w^2}dw=\frac{1}{\sqrt{2}}\ln\left|\frac{w+\sqrt{2}}{w-\sqrt{2}}\right|+C=\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{1-\csc x}+\sqrt{2}}{\sqrt{1-\csc x}-\sqrt{2}}\right|+C$$

187. $t=\tan x$，$dt=\sec^2xdx$，$u=t+\sqrt{1+t^2}$，$t=\frac{1}{2}\left(u-\frac{1}{u}\right)$，$dt=\frac{1}{2}\left(1+\frac{1}{u^2}\right)du$

$$\int\frac{\sec^2x}{(\sec x+\tan x)^{5/2}}dx=\int\frac{1}{(t+\sqrt{1+t^2})^{5/2}}dt=\frac{1}{2}\int(u^{-5/2}+u^{-9/2})du=-\frac{1}{3}u^{-3/2}-\frac{1}{7}u^{-7/2}+C$$

$$=-\frac{1}{3}(\sec x+\tan x)^{-3/2}-\frac{1}{7}(\sec x+\tan x)^{-7/2}+C$$

188. $t=\sec x+\tan x$，$dt=\sec x(\sec x+\tan x)dx$

$$\int\frac{\sec^2x}{(\sec x+\tan x)^{9/2}}dx=\frac{1}{2}\int\frac{\sec x(\sec x+\tan x)+\sec x(\sec x-\tan x)}{(\sec x+\tan x)^{9/2}}dx$$

$$=\frac{1}{2}\int\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{9/2}}dx+\frac{1}{2}\int\frac{\sec x(\sec x-\tan x)}{(\sec x+\tan x)^{9/2}}dx$$

$$=\frac{1}{2}\int\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{9/2}}dx+\frac{1}{2}\int\frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)^{13/2}}dx$$

$$=\frac{1}{2}\int t^{-9/2}dt+\frac{1}{2}\int t^{-13/2}dt$$

$$=-\frac{1}{7}t^{-9/2}-\frac{1}{11}t^{-11/2}+C$$

$$=-\frac{1}{7}(\sec x+\tan x)^{-9/2}-\frac{1}{11}(\sec x+\tan x)^{-11/2}+C$$

189. $\int\sin2022x\cdot\sin^{2020}xdx=\int\sin(2021x+x)\cdot\sin^{2020}xdx$

$$=\int\sin2021x\cdot\cos x\cdot\sin^{2020}xdx+\int\cos2021x\cdot\sin^{2021}xdx$$

$$=\sin2021x\cdot\frac{1}{2021}\sin^{2021}x-\frac{2021}{2021}\int\cos2021x\cdot\sin^{2021}xdx+\int\cos2021x\cdot\sin^{2021}xdx$$

$$= \frac{1}{2021}\sin 2021x \cdot \sin^{2021}x dx + C$$

190. sol 1) $\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$, $t^2 = 1-\tan^2 x$, $2tdt = -2\tan x \sec^2 dx$

$$\int \frac{\sqrt{\cos 2x}}{\sin x}dx = \int \frac{\sqrt{1-\tan^2 x}}{\tan x}dx = \int \frac{t}{\tan x} \cdot \frac{-t}{\tan x \sec^2 x}dt = -\int \frac{t^2}{(1-t^2)(2-t^2)}dt$$

$$= -\int \left(\frac{1}{1-t^2} - \frac{2}{2-t^2}\right)dt = -\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + C$$

$$= -\frac{1}{2}\ln\left|\frac{1+\sqrt{1-\tan^2 x}}{1-\sqrt{1-\tan^2 x}}\right| + \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + C$$

sol 2) $u = \frac{\cos x}{\sqrt{\cos 2x}}$, $du = \frac{-\sin x\sqrt{\cos 2x} + \cos x \cdot \frac{\sin 2x}{\sqrt{\cos 2x}}}{\cos 2x}dx$

$$= \frac{-\sin x \cos 2x + \cos x \sin 2x}{(\sqrt{\cos 2x})^3}dx = \frac{\sin x}{(\sqrt{\cos 2x})^3}dx$$

$$\int \frac{\sqrt{\cos 2x}}{\sin x}dx = \int \frac{\cos^2 2x}{\sin^2 x}du = \int \frac{\cos 2x}{\cos^2 x - \cos^2 x + \sin^2 x} \cdot \frac{\cos 2x}{2\cos^2 x - \cos^2 x + \sin^2 x}du$$

$$= \int \frac{\cos 2x}{\cos^2 x - \cos 2x} \cdot \frac{\cos 2x}{2\cos^2 x - \cos 2x}du = \int \frac{1}{(u^2-1)(2u^2-1)}du$$

$$= \int \left(\frac{1}{u^2-1} - \frac{2}{2u^2-1}\right)du = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}u-1}{\sqrt{2}u+1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{\cos x - \sqrt{\cos 2x}}{\cos x + \sqrt{\cos 2x}}\right| - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos x - \sqrt{\cos 2x}}{\sqrt{2}\cos x + \sqrt{\cos 2x}}\right| + C$$

sol 3) $t = \tan\frac{x}{2}$, $dt = \frac{1}{2}\sec^2\frac{x}{2}dx = \frac{t^2+1}{2}dx$, $u = t^2$, $du = 2tdt$

$y - u = \sqrt{u^2 - 6u + 1}$, $u = \frac{y^2-1}{2y-6}$ (오일러 치환)

$$\int \frac{\sqrt{\cos 2x}}{\sin x}dx = \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x}dx = \int \frac{\sqrt{t^4 - 6t^2 + 1}}{t^3 + t}dt = \frac{1}{2}\int \frac{\sqrt{u^2 - 6u + 1}}{u^2 + u}du$$

$$= \frac{1}{2}\int \frac{(y^2 - 6y + 1)^2}{(y-1)(y-3)(y+1)(y^2 + 2y - 7)}dy$$

$$= \frac{1}{2}\int \left(\frac{1}{y-1} + \frac{1}{y-3} - \frac{1}{y+1} - \frac{16}{(y+1)^2 - 8}\right)dy$$

$$= \frac{1}{2}\ln|y-1| + \frac{1}{2}\ln|y-3| - \frac{1}{2}\ln|y+1| - \sqrt{2}\ln\left|\frac{y - 2\sqrt{2} + 1}{y + 2\sqrt{2} + 1}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(y-1)(y-3)}{y+1}\right| - \sqrt{2}\ln\left|\frac{y - 2\sqrt{2} + 1}{y + 2\sqrt{2} + 1}\right| + C$$

$$\left(y = \tan^2\frac{x}{2} + \sqrt{\tan^4\frac{x}{2} - 6\tan^2\frac{x}{2} + 1}\right)$$

191. 제 1종 오일러 치환 : $\sqrt{x^2 + 4x - 4} = x + t$, $x = \dfrac{t^2 + 4}{4 - 2t}$, $dx = \dfrac{-2t^2 + 8t + 8}{(4-2t)^2}dt$

$$\int \frac{1}{x\sqrt{x^2 + 4x - 4}}dx = \int \frac{4-2t}{t^2 + 4} \cdot \frac{4-2t}{-t^2 + 4t + 4} \cdot \frac{-2t^2 + 8t + 8}{(4-2t)^2}dt = 2\int \frac{1}{t^2 + 4}dt$$

$$= \tan^{-1}\left(\frac{t}{2}\right) + C = \tan^{-1}\left(\frac{\sqrt{x^2 + 4x - 4} - x}{2}\right) + C$$

192. 제 2종 오일러 치환 : $\sqrt{-x^2 + x + 2} = xt + \sqrt{2}$, $x = \dfrac{1 - 2\sqrt{2}t}{t^2 + 1}$, $dx = \dfrac{2\sqrt{2}t^2 - 2t - 2\sqrt{2}}{(t^2+1)^2}dt$

$$\int \frac{1}{x\sqrt{-x^2 + x + 2}}dx = \int \frac{t^2 + 1}{1 - 2\sqrt{2}t} \cdot \frac{t^2 + 1}{-\sqrt{2}t^2 + t + \sqrt{2}} \cdot \frac{2\sqrt{2}t^2 - 2t - 2\sqrt{2}}{(t^2 + 1)^2}dt$$

$$= \int \frac{-2}{-2\sqrt{2}t + 1}dt = \frac{1}{\sqrt{2}}\int \frac{-2\sqrt{2}}{-2\sqrt{2}t + 1}dt = \frac{1}{\sqrt{2}}\ln|2\sqrt{2}t - 1| + C$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{2\sqrt{-2x^2 + 2x + 4} - 4}{x} - 1\right| + C$$

193. 제 3종 오일러 치환 : $\sqrt{-(x-2)(x-1)} = (x-2)t$, $x = \dfrac{-2t^2 - 1}{-t^2 - 1}$, $dx = \dfrac{2t}{(-t^2 - 1)^2}dt$

$$\int \frac{x^2}{\sqrt{-x^2+3x-2}}dx = \int \left(\frac{-2t^2-1}{-t^2-1}\right)^2 \cdot \frac{-t^2-1}{t} \cdot \frac{2t}{(-t^2-1)^2}dt = -2\int \frac{(2t^2+1)^2}{(t^2+1)^3}dt$$

$$= -2\int \left(\frac{4}{t^2+1} - \frac{4}{(t^2+1)^2} + \frac{1}{(t^2+1)^3}\right)dt = -8\tan^{-1}t + 8I - 2J$$

$$= -\frac{19}{4}\tan^{-1}t + \frac{t(13t^2+11)}{4(t^2+1)^2} + C = -\frac{19}{4}\tan^{-1}\left(\frac{\sqrt{-(x-2)(x-1)}}{x-2}\right) - \frac{(2x+9)\sqrt{-(x-1)(x-2)}}{4} + C$$

※ $x = \tan t$, $dx = \sec^2 t dt$, $I = \int \frac{1}{(x^2+1)^2}dx = \int \cos^2 t dt = \frac{1}{2}\int (1+\cos 2t)dt$

$$= \frac{1}{2}t + \frac{1}{4}\sin 2t + C = \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)} + C$$

※※ $x = \tan t$, $dx = \sec^2 t dt$, $J = \int \frac{1}{(x^2+1)^3}dx = \int \cos^4 t dt = \frac{1}{4}\int (1+\cos 2t)^2 dt$

$$= \frac{1}{4}\int (1+2\cos 2t + \cos^2 2t)dt = \frac{1}{4}\int \left(1+2\cos 2t + \frac{1+\cos 4t}{2}\right)dt$$

$$= \frac{1}{8}\int (3+4\cos 2t + \cos 4t)dt = \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + C = \frac{3}{8}\tan^{-1}x + \frac{x(3x^2+5)}{8(x^2+1)^2} + C$$

194. $t = \cos\theta$, $dt = -\sin\theta d\theta$, $u = \frac{t-1}{t+1}$, $t = \frac{1+u}{1-u}$, $dt = \frac{2}{(1-u)^2}du$

$$\int \frac{\sin^3(\theta/2)}{\cos(\theta/2)\cdot\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta = \frac{1}{2}\int \frac{(1-\cos\theta)\sin\theta}{(1+\cos\theta)\sqrt{\cos^3\theta+\cos^2\theta+\cos\theta}}d\theta$$

$$= \frac{1}{2}\int \frac{t-1}{(t+1)\sqrt{t^3+t^2+t}}dt$$

$$= \frac{1}{2}\int u \cdot \frac{\sqrt{(1-u)^3}}{\sqrt{(1+u)^3+(1+u)^2(1-u)+(1+u)(1-u)^2}} \cdot \frac{2}{(1-u)^2}du$$

$$= \int \frac{u}{\sqrt{3-2u^2-u^4}}du = \frac{1}{2}\int \frac{2u}{\sqrt{4-(u^2+1)^2}}du = \frac{1}{2}\sin^{-1}\left(\frac{u^2+1}{2}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{t^2+1}{(t+1)^2}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{\cos^2\theta+1}{(\cos\theta+1)^2}\right)+C = \frac{1}{2}\sin^{-1}\left[\frac{1}{4}\sec^4\left(\frac{\theta}{2}\right)-\tan^2\left(\frac{\theta}{2}\right)\right]+C$$

195. $\displaystyle\int \frac{\tan^4\theta}{1-\tan^2\theta}d\theta = \int \frac{\tan^4\theta-1}{1-\tan^2\theta}d\theta + \int \frac{1}{1-\tan^2\theta}d\theta$

$$= -\int (\tan^2\theta+1)d\theta + \int \frac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}d\theta = -\int \sec^2\theta d\theta + \int \frac{1+\cos 2\theta}{2\cos 2\theta}d\theta$$

$$= -\tan\theta + \int\left(\frac{1}{2}\sec 2\theta + \frac{1}{2}\right)d\theta = -\tan\theta + \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| + \frac{1}{2}\theta + C$$

196. $t = x - \dfrac{1}{x}$, $dt = \left(1+\dfrac{1}{x^2}\right)dx$, $\dfrac{dx}{x} = \dfrac{x}{x^2+1}dt$

$$I = \int \frac{x^2+1}{x^4+3x^3+3x^2-3x+1}dx = \int \frac{x+x^{-1}}{(x^2+x^{-2})+3(x-x^{-1})+3}\cdot\frac{dx}{x}$$

$$= \int \frac{1}{t^2+3t+5}dt = \int \frac{1}{\left(t+\frac{3}{2}\right)^2+\frac{11}{4}}dt = \sqrt{\frac{4}{11}}\tan^{-1}\left(\sqrt{\frac{4}{11}}\left(t+\frac{3}{2}\right)\right)+C$$

$$= \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{2}{\sqrt{11}}\left(x-\frac{1}{x}+\frac{3}{2}\right)\right)+C.$$

cf) 위 표현은 $x=0$에서 불연속이다. 따라서 정적분을 할 때에는 $x>0$인 구간과 $x<0$인 구간의 적분상수를 각각 다르게 설정해야 한다. 이들 적분상수를 각각 C_+, C_-라 하면, 위 적분은 $x=0$에서 well-behaved하므로 그 부정적분 I는 $x=0$에서 연속이다. 즉,

$$C_+ - C_- = \frac{2\pi}{\sqrt{11}}$$

이다. 이를 이용하면

$$I = \frac{2}{\sqrt{11}}\left[\tan^{-1}\left(\frac{2}{\sqrt{11}}\left(x-\frac{1}{x}+\frac{3}{2}\right)\right)+\delta(x)\right]+C'$$

$$\left(C' = \frac{C_+ + C_-}{2},\ \delta(x) = \begin{cases} +\dfrac{\pi}{2} & (x>0) \\ -\dfrac{\pi}{2} & (x<0) \end{cases} \right)$$

이다. 이를 간단히하기 위해

$$\delta(x)=\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right),\ x\neq 0$$

이라 하면,

$$I=\frac{2}{\sqrt{11}}\left[\tan^{-1}\left(\frac{2}{\sqrt{11}}\left(x-\frac{1}{x}+\frac{3}{2}\right)\right)+\tan^{-1}\left(\frac{2}{\sqrt{11}\,x}\right)+\tan^{-1}\left(\frac{\sqrt{11}\,x}{2}\right)\right]+C'$$

$$=\frac{2}{\sqrt{11}}\left[\tan^{-1}\left(\frac{\sqrt{11}\,x^2(2x+3)}{7x^2-6x+4}\right)+\tan^{-1}\left(\frac{\sqrt{11}\,x}{2}\right)\right]+C'$$

이고 이때 I는 $\forall x\in\mathbb{R}$에 대해 연속이다.

197. $t=\sqrt{x+1+x^{-1}}$, $dt=\dfrac{d(x+1+x^{-1})}{2\sqrt{x+1+x^{-1}}}$

$$\int\frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx=\int\frac{x-1}{(x+1)\sqrt{x+1+x^{-1}}}\frac{dx}{x}$$

$$=\int\frac{x-1}{(x+1)\sqrt{x+1+x^{-1}}}\frac{d(x+1+x^{-1})}{x-x^{-1}}=\int\frac{d(x+1+x^{-1})}{(x+2+x^{-1})\sqrt{x+1+x^{-1}}}=2\int\frac{1}{1+t^2}dt$$

$$=2\tan^{-1}t+C=2\tan^{-1}\sqrt{x+1+x^{-1}}+C$$

198. $\displaystyle\int\frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx=\int\left(\frac{e^x}{(1-x)\sqrt{1-x^2}}+e^x\sqrt{\frac{1+x}{1-x}}\right)dx$

$$=\int\left(e^x\left(\sqrt{\frac{1+x}{1-x}}\right)'+(e^x)'\sqrt{\frac{1+x}{1-x}}\right)dx=e^x\sqrt{\frac{1+x}{1-x}}+C$$

199. $\left(\dfrac{u}{v}\right)'=\dfrac{u'v-v'u}{v^2}$ 이므로

$$\int\frac{u'v-v'u}{v^2}dx=\int\left(\frac{u}{v}\right)'dx=\frac{u}{v}+C$$

이다. 이때

$$I=\int\left(\frac{\tan^{-1}x}{1+(x+1/x)\tan^{-1}x}\right)^2dx=\int\frac{x^2(\tan^{-1}x)^2}{(x+(x^2+1)\tan^{-1}x)^2}dx$$

이고, $v=x+(x^2+1)\tan^{-1}x$라 하면 $v'=2+2x\tan^{-1}x$이다. 이제

$$u'v-v'u=u'\left[x+(x^2+1)\tan^{-1}x\right]-u\left[2+2x\tan^{-1}x\right]=x^2(\tan^{-1}x)^2 \quad \cdots \quad [1]$$

인 함수 $u(x)$를 찾으면 된다. 만약

$$u=\frac{-x^2+(x^2+1)(\tan^{-1}x)^2}{2}$$

이면

$$u'=-x+x(\tan^{-1}x)^2+\tan^{-1}x$$

이고 이 u는 기적적으로 [1]식을 만족시킨다. 즉,

$$I=\int\left(\frac{-x^2+(1+x^2)(\tan^{-1}x)^2}{2\left[x+(x^2+1)\tan^{-1}x\right]}\right)'dx=\frac{-x^2+(1+x^2)(\tan^{-1}x)^2}{2\left[x+(x^2+1)\tan^{-1}x\right]}+C.$$

200. $\displaystyle\int\frac{1}{\prod_{i=0}^{m}(x+i)}dx=\int\frac{1}{x(x+1)(x+2)\cdots(x+m)}dx \quad (m\in\mathbb{N}\cup\{0\})$

sol 1) $q_m(x):=\displaystyle\prod_{k=0}^{m}(x+k)$, $f_m(x):=\dfrac{1}{q_m(x)}$라 하자. 실수 A_i $(0\le i\le m)$에 대하여 f_m을 부분분수의 합으로 분해하면

$$f_m(x)=\sum_{n=0}^{m}\frac{A_n}{x+n}$$

이다. 따라서 f_m의 부정적분은

$$\int f_m(x)dx=\sum_{n=0}^{m}A_n\ln|x+n|+C$$

이다. 또한

$$1=f_m(x)q_m(x)=q_m(x)\sum_{n=0}^{m}\frac{A_n}{x+n}=\sum_{n=0}^{m}A_n p_n(x)$$

라 하면,

$$p_n(x) = \prod_{k \neq n}(x+k) = (-1)^n \prod_{k=0}^{n-1}(-x-k) \cdot \prod_{k=n+1}^{m}(k+x)$$

이다. 이때

$$p_n(-n) = (-1)^n \prod_{k=0}^{n-1}(n-k) \cdot \prod_{k=n+1}^{m}(k-n) = (-1)^n n!(m-n)! = \frac{(-1)^n m!}{\binom{m}{n}}$$

이고 $n \neq k \leq m$인 음이 아닌 정수 k에 대하여

$$p_n(-k) = 0$$

이다. 따라서

$$A_n = \frac{1}{p_n(-n)} = \frac{(-1)^n}{m!}\binom{m}{n}$$

이고

$$\int f_m(x)dx = \sum_{n=0}^{m} \frac{(-1)^n}{m!}\binom{m}{n} \ln|x+n| + C.$$

sol 2) $\dfrac{1}{\prod_{i=0}^{m}(x+i)} = \dfrac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{j=0}^{m}\dfrac{a_j}{x+j}$ $\cdots$ [1] 이라 하자.

헤비사이드법에 의해

$$a_j = \lim_{x \to (-j)} \frac{x+j}{x(x+1)(x+2)\cdots(x+m)}$$

이다. 이때

$$a_1 = \frac{1}{(-1)(2)(3)\cdots(m-1)} = \frac{(-1)^1}{(m-1)!},$$

$$a_2 = \frac{1}{(-2)(-1)(1)(2)\cdots(m-2)} = \frac{(-1)^2}{2!(m-2)!},$$

$$a_3 = \frac{1}{(-3)(-2)(-1)(1)\cdots(m-3)} = \frac{(-1)^3}{3!(m-3)!}$$

이고 수학적 귀납법을 이용하면

$$a_j = \frac{(-1)^j}{j!(m-j)!} = \frac{(-1)^j}{m!}\binom{m}{j}$$

임이 증명된다. 따라서 [1]의 양변을 적분하면

$$\int \frac{1}{x(x+1)(x+2)\cdots(x+m)}\,dx = \sum_{j=0}^{m}\int \frac{a_j}{x+j}\,dx,$$

$$\int \frac{1}{x(x+1)(x+2)\cdots(x+m)}\,dx = \sum_{j=0}^{m} a_j \ln|x+j| + C$$

$$= \sum_{j=0}^{m}\frac{(-1)^j}{m!}\binom{m}{j}\ln|x+j| + C.$$